A SPECIAL CONGRUENCE

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1. It is familiar that if $p$ is a prime such that $p \nmid m$, $p^r \nmid m$ then
\[(1.1) \quad B_m \equiv 0 \pmod{p^r}, \]
where $B_m$ denotes a Bernoulli number in the even suffix notation. The writer has recently proved the companion formula ([2, Theorem 3]; see also [1])
\[(1.2) \quad B_{m(p-1)} + 1/p - 1 \equiv 0 \pmod{p^r} \quad (p \geq 3), \]
for $p^r \mid m$, $m > 0$; moreover if $m = p^r h$, then
\[(1.3) \quad p^{-r} (B_{m(p-1)} + 1/p - 1) \equiv h w_p \pmod{p} \quad (p > 3), \]
where $w_p$ denotes Wilson's quotient $((p - 1)! + 1)/p$.

In this note we show that the above formulas imply
\[(1.4) \quad p + (p - 1) \sum_{0 < s(p-1) < m} \left( \frac{m}{s(p - 1)} \right) \equiv 0 \pmod{p^{r+1}}, \]
where $p^r \mid m$ and $p \geq 3$. More precisely if $m = p^r m_0$, we have, for $p > 3$,
\[(1.5) \quad p^{r-1} \left\{ p + (p - 1) \sum_{0 < s(p-1) < m} \left( \frac{m}{s(p - 1)} \right) \right\} \equiv m_0 \left\{ \frac{1}{2} - \sum_{0 < 2s < m, p - 1 \nmid 2s} \left( \frac{m - 1}{2s - 1} \right) \frac{B_{2s}}{2s} + \delta_m \frac{w_p}{p - 1} \right\} \pmod{p}, \]
where $\delta_m = 1$ for $p - 1 \mid m - 1$, $\delta_m = 0$ otherwise.

For $r = 0$, (1.4) is due to Hermite. The proof below of (1.4) was suggested by Nielsen's proof [3, p. 254] of Hermite's formula.

2. Proof of (1.4). Using the basic recurrence for the Bernoulli numbers we may write
\[(2.1) \quad 1 - \frac{1}{2} m + \sum_{0 < 2s < m} \left( \frac{m}{2s} \right) B_{2s} = 0. \]

Now let $p^r \mid m$. Consider first a term such that $p - 1 \nmid 2s$. Let $p^k \mid s$, so that by (1.1), $B_{2s} \equiv 0 \pmod{p^k}$. If $k \leq r$, it follows that
\[(2.2) \quad \left( \frac{m}{2s} \right) = \frac{m}{2s} \left( \frac{m - 1}{2s - 1} \right) \equiv 0 \pmod{p^{r-k}}. \]

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and consequently

\[(2.3) \quad \binom{m}{2s} B_{2s} \equiv 0 \pmod{p^r}.\]

Clearly (2.3) holds also for \( k > r \). Thus (2.1) and (2.2) imply

\[
1 + \sum_{0 < s < (p-1) < m} \binom{m}{s(p - 1)} B_{s(p-1)} \equiv 0 \pmod{p^r},
\]

which may be rewritten as

\[
1 + \sum_{0 < s < (p-1) < m} \binom{m}{s(p - 1)} \left(B_{s(p-1)} + \frac{1}{p} - 1\right) \equiv \left(\frac{1}{p} - 1\right) \sum_{0 < s < (p-1) < m} \binom{m}{s(p - 1)} \pmod{p^r}.
\]

Now exactly as in proving (2.3), we may show, using (1.2), that

\[
\binom{m}{s(p - 1)} \left(B_{s(p-1)} + \frac{1}{p} - 1\right) \equiv 0 \pmod{p^r}.
\]

Thus (2.4) reduces to

\[(2.5) \quad 1 \equiv \left(\frac{1}{p} - 1\right) \sum_{0 < s < (p-1) < m} \binom{m}{s(p - 1)} \pmod{p^r}.
\]

It is evident that (2.5) and (1.4) are equivalent.

3. Proof of (1.5). We again begin with (2.1) which we now write as

\[
1 - \frac{1}{2} m + \sum_{0 < s < m, p-1 \nmid s} \binom{m}{2s} B_{2s} + \sum_{0 < s < (p-1) < m} \binom{m}{s(p - 1)} \left(B_{s(p-1)} + \frac{1}{p} - 1\right) = 0.
\]

This evidently implies

\[
1 - \frac{1}{2} m + \sum_{0 < s < m, p-1 \nmid s} \binom{m}{2s} B_{2s} + \sum_{0 < s < (p-1) < m} \binom{m}{s(p - 1)} \left(B_{s(p-1)} + \frac{1}{p} - 1\right)
\]

\[(3.1) \quad = \sum_{0 < s < (p-1) < m} \binom{m}{s(p - 1)} \left(\frac{1}{p} - 1\right).
\]
Consider first the sum
\[(3.2) \quad S = \sum_{0 < s < (p-1) < m} \left( \binom{m}{s(p - 1)} \right) \left( B_{s(p-1)} + \frac{1}{p} - 1 \right).\]

Let \( p^k | s \) and put
\[ s = p^k h; \]

then by (1.3) we have
\[(3.3) \quad B_{s(p-1)} + \frac{1}{p} - 1 \equiv p^k h w_p \pmod{p^{k+1}}.\]

If \( k \leq r \) it is evident from (2.2) that (3.3) yields
\[
\left( \binom{m}{s(p - 1)} \right) \left( B_{s(p-1)} + \frac{1}{p} - 1 \right) \equiv \left( \binom{m}{s(p - 1)} \right) p^k h w_p \pmod{p^{k+1}};
\]
clearly (3.4) holds also for \( k > r \). Since the right member of (3.4) is equal to
\[ m \left( \binom{m - 1}{s(p - 1) - 1} \right) w_p / (p - 1), \]
we see that (3.2) becomes
\[(3.5) \quad S \equiv \frac{mw_p}{p - 1} \sum_{0 < s < (p-1) < m} \left( \binom{m - 1}{s(p - 1) - 1} \right) \pmod{p^{r+1}}.\]

In the next place for the first sum in the left member of (3.1) we have
\[(3.6) \quad \sum_{0 < 2s < m, p - 1 \frac{1}{2}s} \left( \binom{m}{2s} \right) B_{2s} = \sum_{0 < 2s < m, p - 1 \frac{1}{2}s} \left( \frac{m - 1}{2s - 1} \right) \frac{B_{2s}}{2s}.\]

Substituting from (3.5) and (3.6) in (3.1) we get
\[
1 - \left( \frac{1}{p} - 1 \right) \sum_{0 < s < (p-1) < m} \left( \binom{m}{s(p - 1)} \right) \equiv \frac{1}{2} \left( m - m \sum_{0 < 2s < m, p - 1 \frac{1}{2}s} \left( \frac{m - 1}{2s - 1} \right) \frac{B_{2s}}{2s} \right.
\]
\[
- \frac{mw_p}{p - 1} \sum_{0 < s < (p-1) < m} \left( \binom{m - 1}{s(p - 1) - 1} \right) \pmod{p^{r+1}}.
\]
Now let \( p^r | m, p^{r+1} \nmid m \); then (3.7) becomes

\[
\frac{1}{m} \left\{ 1 - \left( \frac{1}{p} - 1 \right) \sum_{0 \leq s < (p-1) < m} \left( \frac{m}{s(p-1)} \right) \right\}
\]

\[
= \frac{1}{2} - \frac{w_p}{\phi - 1} \sum_{0 < 2s < m, \phi - 1 | 2s} \frac{B_{2s}}{2s} - \frac{\phi - 1}{\phi - 1} \sum_{0 < s < (p-1) < m} \left( \frac{m-1}{s(p-1) - 1} \right) \pmod{p}.
\]

But [3, p. 255]

\[
\sum_{0 < s < (p-1) < m} \left( \frac{m-1}{s(p-1) - 1} \right) = \begin{cases} 0 & (\phi - 1 | m - 1), \\
-1 & (\phi - 1 | m - 1);
\end{cases}
\]

indeed (3.9) is an easy consequence of the case \( r = 0 \) of (1.4). Finally (3.7), (3.8), and (3.9) evidently imply (1.5).

\section*{References}

