ON ALGEBRAS OF CONTINUOUS FUNCTIONS

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Let C denote the algebra of functions continuous on the unit circle. With norm $\|f\| = \sup_{|\lambda|=1} |f(\lambda)|$, C is a Banach algebra. Let A denote the set of all f in C which are boundary values of functions analytic in $|z| < 1$ and continuous in $|z| \leqq 1$. A is then a closed subalgebra of C, and by known results A consists of those and only those f in C for which $\int_{|\lambda|=1} f(\lambda)\lambda^n d\lambda = 0$, $n \geqq 0$.

In [1] the question was raised whether if $\phi(\lambda)$ is any function in C and not in A, the closed algebra generated by ϕ and A is all of C. It was shown in [1] that if ϕ is real or if ϕ satisfies a Lipschitz condition, then the algebra generated by A and ϕ does equal C. In the following theorem the question is answered in the affirmative.

THEOREM 1. *If A' is any closed subalgebra of C which includes A, then either $A' = A$ or $A' = C$.*

Our proof does not make use of the results of [1]. Our chief tool is the following known theorem (see [2, pp. 19–21; 3, p. 162]). Let $d\mu(\lambda)$ be a complex-valued measure on the circle such that $\int_{|\lambda|=1} \lambda^n d\mu(\lambda) = 0$, $n \geqq 0$. Then there exists a function $h(z)$ analytic in $|z| < 1$ with the following properties: $h(e^{i\theta}) = \lim_{r\to 1} h(re^{i\theta})$ exists for almost all θ and is summable on $[0, 2\pi]$ and $\lim_{r \to 1} \int_0^{2\pi} |h(re^{i\theta}) - h(e^{i\theta})| d\theta = 0$ and $d\mu(\lambda) = h(\lambda)d\lambda$, i.e. if f is in C, $\int_{|\lambda|=1} f(\lambda) d\mu(\lambda) = \int_{|\lambda|=1} f(\lambda)h(\lambda)d\lambda$. For a given θ, moreover, $h(z)$ approaches $h(e^{i\theta})$ along any nontangential path.

PROOF OF THEOREM 1. Suppose $A' \neq C$. Let $\phi(\lambda)$ be any function in A'. We claim that ϕ is in A.

For since A' is a proper closed subspace of C, there exists a linear functional on C which is not null but which annihilates every f in A' and which hence annihilates every function $\lambda^n\phi^m(\lambda)$ with $n, m \geqq 0$. By the representation of linear functionals on C, there thus exists a complex measure $d\mu(\lambda)$ with

$$\int_{|\lambda|=1} \lambda^n \phi^m(\lambda) d\mu(\lambda) = 0, \qquad n, m \geqq 0.$$

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In particular, for $m=0$, we get $\int_{|\lambda|=1}\lambda^n d\mu(\lambda)=0$. As we noted, this implies the existence of an analytic function $h(z)$ with $d\mu(\lambda)=h(\lambda)d\lambda$. Fix now m and consider the measure $\phi^m(\lambda)h(\lambda)d\lambda$ on $|\lambda|=1$. This measure annihilates λ^n for $n\geqq 0$ and by the same theorem there exists an analytic function $\gamma_m(z)$ with $\phi^m(\lambda)h(\lambda)=\gamma_m(\lambda)$.

Set now $\gamma_1(\lambda)=g(\lambda)$. Then $g^m(\lambda)=\phi^m(\lambda)h^m(\lambda)=h^{m-1}(\lambda)\gamma_m(\lambda)$ for $|\lambda|=1$. But $h^{m-1}\cdot\gamma_m$ and g^m are analytic functions and since their (nontangential) boundary values are equal almost everywhere, the functions themselves are equal, i.e. $g^m(z)=h^{m-1}(z)\gamma_m(z)$ for $|z|<1$. The last statement follows at once from a theorem of Priwaloff [4] which asserts that if an analytic function has nontangential boundary values on $|\lambda|=1$ vanishing on a set of positive measure, then the function vanishes identically.

We now assert that if h has a zero of order p at α, $|\alpha|<1$, then g has at α a zero of order $q\geqq p$. For since $g^m(z)=h^{m-1}(z)\gamma_m(z)$ and γ_m is analytic at α, we have $mq\geqq(m-1)p$. Since this holds for arbitrarily large m, we get $q\geqq p$ as asserted. The function $g(z)/h(z)$ is then analytic in $|z|<1$. Also, this function has $\phi(e^{i\theta})$ as boundary value a.e. Thus for any ϕ in A' there exists a function $\Phi(z)$ analytic in $|z|<1$ and taking $\phi(e^{i\theta})$ as (nontangential) boundary value a.e. By Priwaloff's theorem, then, ϕ determines Φ uniquely. If $\phi_1, \phi_2\in A'$ and Φ_1, Φ_2 take boundary values ϕ_1 and ϕ_2 respectively, then $\Phi_1\cdot\Phi_2$ is the unique analytic function with boundary value $\phi_1\cdot\phi_2$, and similarly for $\Phi_1+\Phi_2$. Fix now z_0, $|z_0|<1$. The mapping $\phi\rightarrow\Phi(z_0)$ is thus a multiplicative linear functional defined on the Banach algebra A'. Hence $|\Phi(z_0)|\leqq\|\phi\|$. Since z_0 was arbitrary, each $\Phi(z)$ is bounded in $|z|<1$.

Take now any ϕ in A' and let Φ be the corresponding analytic function. Then the functions $\Phi_r(\theta)=\Phi(re^{i\theta})$ are uniformly bounded as r approaches 1. Also $\phi(e^{i\theta})=\lim_{r=1}\Phi_r(\theta)$ a.e. Hence $\int_{|\lambda|=1}\lambda^n\phi(\lambda)d\lambda$ $=\lim_{r=1}\int_{|\lambda|=r}\lambda^n\Phi(\lambda)d\lambda=0$, $n\geqq 0$.

Hence ϕ is in A, and the theorem is proved.

It is of interest to note the following concerning the proof we have given: In order that a function $\phi(\lambda)$ in C belong to A it is not sufficient that there exist a function $\Phi(z)$ analytic in $|z|<1$ and with nontangential boundary values equal to $\phi(e^{i\theta})$ a.e. Let $\Phi(z)$ $=(1-z)e^{(1+z)/(1-z)}$. Then $\lim_{z\rightarrow e^{i\theta}}\Phi(z)=(1-e^{i\theta})e^{i\cot(\theta/2)}=\phi(\theta)$ a.e. Now $\phi(\theta)$ is in C, and $\Phi(z)$ is even the quotient of two bounded analytic functions. Still $\phi(\theta)$ is not in A.

THEOREM 2. *Let B be any closed subalgebra of C which contains one function $\phi(\lambda)$ such that $\phi(\lambda_1)=\phi(\lambda_2)$ implies $\lambda_1=\lambda_2$. Then if B' is any closed subalgebra of C which includes B, either $B'=B$ or $B'=C$.*

PROOF. Let Γ be the simple closed Jordan curve which is the image of the unit circle under the function $\phi(\lambda)$. Let $\gamma(z)$ be the function which maps the interior of the unit circle conformally on the interior of Γ. Then γ is continuous and one-to-one up to the boundary and so the map: λ into $\gamma(\lambda)$ is a homeomorphism of the circle $|\lambda|=1$ on Γ. Let us now define the following mapping τ of C into C: If $f(\lambda)$ is in C, $\tau f(\lambda)=f(\phi^{-1}\gamma(\lambda))$. Clearly τ is an isomorphism of C onto itself. Further $\tau\phi(\lambda)=\gamma(\lambda)$ and so τB, the image of B under τ, includes $\gamma(\lambda)$. Since polynomials in $\gamma(\lambda)$ are dense in A, τB includes A. Let now B' include B. Then $\tau B'$ includes τB and hence $\tau B'$ includes A. By Theorem 1, then, either $\tau B'=C$ or $\tau B'=A$. If $\tau B'=C$, then $B'\equiv C$, since τ is an automorphism. If $\tau B'=A$, then $\tau B'$ is included in τB and so $\tau B'=\tau B$. Hence $B'=B$. q.e.d.

Theorem 1 is a special case of Theorem 2, since A contains the function $\phi(\lambda)=\lambda$. As a corollary of Theorem 1, we also obtain the following result due to Rudin [5]: *Let M be an algebra of functions continuous in the closed disk $|z|\leqq 1$. Suppose every F in M attains the maximum of its modulus on the boundary $|z|=1$. Then if one schlicht analytic function belongs to M, every F in M is analytic in $|z|<1$.*

PROOF. Let $\overline{M}$ be the closure of M in the sense of uniform approximation on $|z|\leqq 1$. Since M contains one schlicht function, $\overline{M}$ contains all analytic functions continuous in $|z|\leqq 1$. Also $\overline{M}$ is an algebra and if F is in $\overline{M}$, $|F(z)|\leqq \sup_{|\lambda|=1}|F(\lambda)|$ for all z, $|z|\leqq 1$.

Let A' be the restriction of $\overline{M}$ to $|z|=1$. Then A' is a subalgebra of C and A' is closed since $\overline{M}$ is closed and also for any sequence f_n in A', $\sup_{|z|\leqq 1}|f_n(z)-f_m(z)|\leqq \sup_{|\lambda|=1}|f_n(\lambda)-f_m(\lambda)|$. Suppose $A'=C$. Then the functional which assigns to each f in C the number $F(0)$, where F is the unique function in $\overline{M}$ with boundary value f, is a multiplicative functional on C and so must equal $f(\lambda_0)$ for some λ_0, $|\lambda_0|=1$. Thus for all F in $\overline{M}$, $F(0)=F(\lambda_0)$, which is false since $F(\lambda)=\lambda$ is in $\overline{M}$. Thus $A'\neq C$.

Now A' includes A, since $\overline{M}$ contains all analytic functions continuous in $|z|\leqq 1$. By Theorem 1, then, $A'=A$, and so if F is in $\overline{M}$ there exists F_1 analytic with $F_1(\lambda)=F(\lambda)$ if $|\lambda|=1$. Since F_1 belongs to $\overline{M}$, it follows that $F_1(z)=F(z)$ for all z, $|z|\leqq 1$. q.e.d.

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SOLUTION OF BERNSTEIN'S APPROXIMATION PROBLEM[1]

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In his famous monograph on approximation theory [2], S. Bernstein initiated the study of the closure properties of sets of functions $\{u^nK(u)\}_0^\infty$ on the real line. It is supposed that $K(u)$ is continuous on $(-\infty, \infty)$ and that $u^nK(u)$ vanishes at $u=\pm\infty$ for each value of n. The problem is to decide when the set $\{u^nK(u)\}$ is fundamental in the space C_0 of functions continuous on $(-\infty, \infty)$, vanishing at $\pm\infty$, and normed by $\|f\|=\max|f(u)|$. So far no necessary and sufficient conditions have been given. A recent paper of Carleson [3] reviews most of the known results, but the paper [1] which seems to come closest to the true conditions has been overlooked.

It is the purpose of this note to give a complete solution. It applies to either real- or complex-valued functions and may be read either way.

THEOREM. *In order that $\{u^nK(u)\}_0^\infty$ be fundamental in C_0 it is necessary and sufficient that*

$$K(u) \neq 0, \qquad -\infty < u < \infty; \tag{1}$$

$$\int_{-\infty}^{\infty} \frac{\log |K(u)|}{1+u^2}\,du = -\infty; \tag{2}$$

and that there exists a sequence of polynomials p_n such that

$$\lim_{n\to\infty} p_n(u)K(u) = 1; \quad |p_n(u)K(u)| \leq C, \quad -\infty < u < \infty. \tag{3}$$

1. **The necessity.** The necessity of (1) is obvious and of (2) is well known [1; 3]. To prove the necessity of the remaining conditions let $0_n(u)$ denote the continuous function which is unity on $(-n, n)$, vanishes outside $(-n-1, n+1)$, and is linear in the remaining in-

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