ON THE ROOTS OF EQUATION $f(x)=\xi$ WHERE $f(x)$ IS REAL AND CONTINUOUS IN (a, b) BUT MONOTONIC IN NO SUBINTERVAL OF (a, b)

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In what follows we always assume that $f(x)$ is a real function for $a \leqq x \leqq b$. Let m and M denote the greatest lower bound and the least upper bound respectively of $f(x)$ when $a \leqq x \leqq b$. We denote by $S(\xi)$ the set of values of x for which $f(x)=\xi$, $a \leqq x \leqq b$, $m \leqq \xi \leqq M$. It is known [4] that there exists a continuous function $f(x)$ for which $S(\xi)$ has the power of the continuum for all ξ for which $m \leqq \xi \leqq M$. S. Minakshisundaram [2] proved that:

If $f(x)$ is continuous but nondifferentiable, then $S(\xi)$ has the power of the continuum for almost all ξ for which $m \leqq \xi \leqq M$.

I prove a somewhat similar result for the more general class of functions $f(x)$ which are continuous but monotonic in no interval. We know that a nondifferentiable function is not monotonic in any interval, but if a function $f(x)$ is monotonic in no interval then it is not necessarily true that $f(x)$ is nondifferentiable. In fact we have, according to a theorem of Köpcke [1]:

There exists a continuous function $f(x)$ which is monotonic in no interval and yet is differentiable for every value of x.

The theorem that I prove is as follows:

THEOREM 1. *If $f(x)$ is continuous but monotonic in no interval, then $S(\xi)$ has the power of the continuum for a set of values of ξ which is of the second category.*

We use the following lemmas in the proof of this theorem.

LEMMA 1. *If $f(x)$ is continuous in (a, b), then the set T of the values of $f(x)$ at its points of relative maxima is enumerable.*

This is a known result [3, p. 193].

Next let I be any subinterval of (a, b), $S(I, \xi)$ the set of roots of the equation $f(x)=\xi$, $x \in I$, and $L(I)$ the set of values of ξ for which the equation $f(x)=\xi$ has one and only one root in I. Let m_0, M_0 be the lower and upper bounds of $f(x)$ for x in I. We shall now prove

LEMMA 2. *If $f(x)$ satisfies the same conditions as in Theorem 1, then $L(I)$ is nondense in (m_0, M_0).*

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PROOF. Plainly it is sufficient to prove that if $J=(\alpha, \beta)$ is a subinterval of (m_0, M_0), then J contains a subinterval free from the points $L(I)$. If we suppose that $L(I)$ contains a whole subinterval of (m_0, M_0), say (γ, δ), and $f(x_0)=\gamma$, $f(x_1)=\delta$, $x_0, x_1 \in I$, then plainly every line $y=t$, $\gamma \leqq t \leqq \delta$ cuts the curve $y=f(x)$ in one and only one point of the interval (x_0, x_1). Since $f(x)$ is continuous this would imply that $f(x)$ is monotonic in (x_0, x_1) which is contrary to our hypothesis, and hence there is a number ξ such that $\alpha<\xi<\beta$, $\xi \notin L(I)$. Let p, q be two values of x in I for which $f(x)=\xi$ and m_1 and M_1 be respectively the lower and upper bounds of $f(x)$ for x in (p, q). Since $f(x)$ is monotonic in no subinterval of I, it follows that $f(x)$ is not equal to a constant when $p \leqq x \leqq q$ and hence $m_1<M_1$, and either $m_1<\xi$ or $\xi<M_1$. Taking the first case, if t is such that $m_1<t\leqq\xi$, then since $f(x)$ is continuous, $f(p)=f(q)=\xi$, and there exists an x such that $p<x<q$, $f(x)=m_1$, we see that the line $y=t$ meets the curve $y=f(x)$ in at least two points whose x-coordinates are in (p, q); a similar conclusion holds if $\xi<M_1$, $\xi\leqq t<M_1$. Hence $L(I)$ does not contain any point in the open interval (m_1, M_1). But ξ is an interior point of J as well as a point of (m_1, M_1). Hence J and (m_1, M_1) have a common subinterval J_1. This shows that $L(I)$ is not dense in J. This proves Lemma 2.

PROOF OF THEOREM 1. In order to prove the theorem it is enough to show that, if

$$N_0 \subset N_1 \subset N_2 \subset \cdots$$

is a sequence of nondense sets, and N is the outer limiting set of this sequence, then there is a $\xi \notin N$ and such that $S(\xi)$ has the power of the continuum. Let I_0 be an interval of values of x and let M_0, m_0 be the upper and lower bounds of $f(x)$ in I_0. We denote by CX the set that is complementary to X. Let J_0 be a closed interval contained in both CN_0 and (m_0, M_0). By Lemma 2, J_0 contains an interval K_0 such that $S(I_0, t)$ contains two or more points for every t in K_0. Plainly there exists a t_0 interior to K_0 and such that $t_0 \notin T$. Since t_0 is in K_0, the line $y=t_0$ meets the curve $y=f(x)$ in at least two points p_0, p_1 whose coordinates are, say, (x_0, t_0), (x_1, t_0) where $x_0, x_1 \in I_0$. Let L_0, L_1 be disjoint closed intervals contained in I_0, each not more than half the length of I_0, and with x_0, x_1 respectively as interior points. Since $t_0 \notin T$, there exists $\delta_0>0$ such that every line $y=t$, $t_0 \leqq t \leqq t_0+\delta_0$, cuts the curve $y=f(x)$ at least once in each of L_0, L_1. We may suppose that δ_0 is so small that the interval $(t_0, t_0+\delta_0)$ is contained in J_0. We can take in $(t_0, t_0+\delta_0)$ a subinterval J_1 such that $J_1 \subset CN_1$, this being possible since N_1 is nowhere dense. In view of

Lemma 2, there is in J_1 a closed sub-interval $(t_0+\delta_1, t_0+\delta_2)$, which we shall call K_1, such that every line $y=t$, $t_0+\delta_1 \leqq t \leqq t_0+\delta_2$ meets the curve $y=f(x)$ in at least two points in each of the intervals L_0, L_1. In the interior of K_1 there is a t_1 such that $t_1 \notin T$. Then plainly t_1 is not a relative maximum of $f(x)$ and the line $y=t_1$ meets the curve $y=f(x)$ in at least four points, two of which have their x-coordinates in the interior of L_0 and two others have their x-coordinates in the interior of L_1. Further, t_1 is contained in the closed interval K_1 which is in both CN_0 and CN_1. Plainly this argument can be repeated and we shall have, at the next stage, a t_2 satisfying the following conditions: (i) t_2 is contained in a closed interval which is in J_1 as well as in CN_2 and therefore also in CN_0 and CN_1. (ii) The line $y=t_2$ cuts the curve $y=f(x)$ in at least eight points, two of which have their x-coordinates in the interior of L_{00}, another two in L_{01}, another two in L_{10}, and another two in L_{11}, where L_{00}, L_{01}, L_{10}, L_{11} are disjoint, each not more than one-fourth the length of I_0, the first two being contained in L_0 and the remaining two in L_1. (iii) $t_2 \notin T$. Proceeding in this way we get $t_3, t_4, \cdots$ converging to a limit $\xi \notin N$. Let $0<\alpha<1$ be a real number whose representation in the dyadic scale is $.\alpha_1\alpha_2\alpha_3 \cdots$ (when there are two representations we choose that one having an infinity of 1's). Plainly $L_{\alpha_1}, L_{\alpha_1\alpha_2}, L_{\alpha_1\alpha_2\alpha_3}, \cdots$ is a nested sequence of intervals having one and only one point x_α in common. From the manner of choice of the intervals $L_0, L_1, L_{00}, L_{01}, L_{10}, L_{11}, L_{000}, \cdots$ it is not difficult to see that $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$. Now $t_n \rightarrow \xi$ as $n \rightarrow \infty$ and the line $y=t_n$ meets the curve $y=f(x)$ in at least one point whose x-coordinate is in $L_{\alpha_1\alpha_2 \cdots \alpha_n}$. As $f(x)$ is continuous, we conclude that $f(x_\alpha)=\xi$. Since to every α there corresponds an x_α different for different values of α it follows that $S(\xi)$ has the power of the continuum. As already observed, $\xi \notin N$ and this completes the proof of Theorem 1.

References

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