ON THE SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Let \( P(x) \) and \( Q(x) \) be complex-valued Lebesgue-measurable functions defined for all non-negative \( x \), the functions \( 1/P(x) \) and \( Q(x) \) being of the class \( L(0, R) \) for every positive \( R \). A solution of the differential equation

\[
(P(x)W')' + Q(x)W = 0
\]

is an absolutely continuous function \( W(x) \) such that \( P(x)W'(x) \) is equal almost everywhere to an absolutely continuous function \( W_1(x) \), say, and that

\[
W_1(x) + Q(x)W = 0
\]

is satisfied for almost all \( x \). In the sequel only those solutions which are distinct from the trivial solution (\( =0 \)) shall be considered.

On the positive \( x \)-axis let \( I \) be an interval which need not be closed or bounded. The equation (1.1) will be called disconjugate on \( I \) if and only if no solution of (1.1) possesses more than one zero on \( I \).

It is the purpose of this note to derive a general criterion (Theorem 1) for the differential equation (1.1) disconjugate on an interval and from which to prove a comparison theorem (Theorem 2). These results generalize those obtained previously by the author for the case \( P(x) = 1 \) [2, Theorems 1 and 9]. When \( P(x) = 1 \) and \( Q(x) \) is real, an interesting discussion of disconjugate differential equations was given by A. Wintner [4].

The method of proof of Theorem 1 is a modification of that employed in [2, Theorem 1].

2. Write

\[
P(x) = p_1(x) + ip_2(x), \quad Q(x) = q_1(x) + iq_2(x),
\]

where \( p_1, p_2, q_1 \) and \( q_2 \) are real. We first prove the following general criterion.

**Theorem 1.** Suppose that the following conditions are satisfied:

1. \( m = m(x) \) is a real-valued function absolutely continuous on every closed subinterval of \( I \),
2. for some real constants \( j \) and \( k \), \( jp_1 + kp_2 \) is positive on \( I \) and \( 1/(jp_1 + kp_2) \) belongs to the class \( L \) on every closed subinterval of \( I \),

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(3) \( m(x) \) satisfies the inequality

\[
(2.2) \quad m' + m^2/(jp_1 + kp_2) \leq -(jq_1 + kq_2)
\]

almost everywhere on \( I \).

Then (1.1) is disconjugate on \( I \). Furthermore, if \( I \) is closed at least at one end, there is a solution of (1.1) which does not vanish on \( I \).

Proof. Suppose that the theorem is not true. Then there is a solution \( W(x) \) which has at least two zeros \( a \) and \( b \), \( a < b \), in \( I \). We shall show that this leads to contradiction.

Let \( W_1 \) be the absolutely continuous function which is equal to \( PW' \) almost everywhere on \( I \). Write

\[
(2.3) \quad W = u + iv, \quad W_1 = u_1 + iv_1,
\]

where \( u, v, u_1, \) and \( v_1 \) are real. It is clear that

\[
(2.4) \quad u_1 = p_1u' - p_2v', \quad v_1 = p_2u' + p_1v'.
\]

Separating the real and imaginary parts of (1.2), we get

\[
(2.5) \quad u' = -q_1u + q_2v, \quad v' = -q_2u - q_1v.
\]

The equalities in (2.4) and (2.5) hold almost everywhere on \( I \). Let

\[
(2.6) \quad L = juu_1 + vv_1 + k(uv_1 - u_1v) - m(u^2 + v^2).
\]

Differentiating (2.6) and simplifying the result with (2.4) and (2.5), we have

\[
(2.7) \quad L' = (jp_1 + kp_2)(u'^2 + v'^2) - 2m(uu' + vv')
\]

almost everywhere on \( I \). Completing the squares, (2.7) yields

\[
L' = (jp_1 + kp_2)[(u' - mu/(jp_1 + kp_2))^2
\]

\[
+ (v' - mv/(jp_1 + kp_2))^2]
\]

\[- [m' + m^2/(jp_1 + kp_2) + jq_1 + kq_2](u^2 + v^2).
\]

The first term on the right-hand side of (2.8) is positive almost everywhere on \([a, b]\), otherwise \( u \) and \( v \) would be solutions of the differential equation

\[
(2.9) \quad y' = my/(jp_1 + kp_2)
\]

on \([a, b]\), and, since \( u \) and \( v \) vanish at \( a \), \( u \) and \( v \) must vanish identically on \([a, b]\), but this is impossible owing to the fact that \( W \neq 0 \). Integrating both sides of (2.8) from \( a \) to \( b \) and using (2.2), we have clearly
Since $L$ vanishes at $a$ and $b$, we have contradiction. This proves that $W$ cannot possess two zeros on $I$ and hence (1.1) is disconjugate on $I$.

If $I$ is closed at the left end with end point $a$, then the argument above shows that $L(x) \geq L(a)$ for all $x$ on $I$. Since $jp_1 + kp_2$ is positive, $j$ and $k$ cannot both be zero. Suppose that $j$ is not zero. Let $W$ be a solution with

$$W(a) = 1, \quad W'(a) = \frac{m(a) + 1}{j}.$$

For this solution it is easy to verify that $L(a) = 1$. Hence $L(x) \geq 1$ for all $x$ on $I$. Consequently, from (2.6), this solution does not vanish on $I$. The cases that $j = 0$, $k \neq 0$, and $I$ is closed at the right end can be proved similarly. This completes the proof of Theorem 1.

3. In this section, we shall prove a comparison theorem. Consider another differential equation

$$(3.1) \quad (r(x)y')' + f(x)y = 0,$$

where $r$ and $f$ are real-valued functions defined for all non-negative $x$, $r$ being positive, and $1/r$ and $f$ belonging to $L(0, R)$ for every positive $R$. On the positive $x$-axis, let $I_0$ be an interval which is either closed or open, and if open need not be bounded.

**Theorem 2.** Suppose that the following conditions are satisfied:

1. $(3.1)$ is disconjugate on $I_0$,
2. for some real constants $j$ and $k$, the inequalities $jp_1 + kp_2 \geq r$, $jq_1 + kq_2 \leq f$ hold almost everywhere on $I_0$.

Then (1.1) is disconjugate on $I_0$. Furthermore, if $I_0$ is closed, there is a solution of (1.1) which does not vanish on $I_0$.

**Proof.** It is known that if $(3.1)$ is disconjugate on $I_0$, there exists a real-valued function $m(x)$ which is absolutely continuous on every closed subinterval of $I_0$ and satisfying the inequality

$$(3.2) \quad m' + m^2/r \leq -f$$

almost everywhere on $I_0$ [3, Theorem 1]. From (3.2) and condition (2) of the theorem, it is clear that

$$(3.3) \quad m' + m^2/(jp_1 + kp_2) \leq -(jq_1 + kq_2)$$

holds almost everywhere on $I_0$. The theorem then follows from Theorem 1.

4. In the following theorem, we consider the differential equation
**Theorem 3.** Suppose that the following conditions are satisfied:

1. $j, k$ and $A$ are real constants,
2. $g = g(x)$ is real-valued, non-negative on $[a, b]$ and belongs to $L(a, b)$,
3. $jp_1 + kp_2$ is positive on $[a, b]$ and $(jp_1 + kp_2)^{-1}$ belongs to $L(a, b)$,
4. (4.1) is disconjugate on $[a, b]$.

Then (1.1) is disconjugate on $[a, b]$.

**Proof.** Since (4.1) is disconjugate on $[a, b]$, according to [3, Theorem 1], there exists a real-valued function $n(x)$ absolutely continuous on $[a, b]$ and satisfying

\[ n' + n^2/(jp_1 + kp_2) \leq -G^2/(jp_1 + kp_2) \]

almost everywhere on $[a, b]$. Let $m = (n - G)/2$. Using (4.2) and (4.3), it is easy to verify that $m$ satisfies (2.2) almost everywhere on $[a, b]$. The theorem then follows from Theorem 1.

Theorem 3 can be easily modified to apply to an open interval, bounded or unbounded.

Theorem 3 is a generalization of a theorem due to P. Hartman [1].

**References**


*The Catholic University of America*