THE NUMBER OF SUBGROUPS OF GIVEN INDEX
IN NONDENUMERABLE ABELIAN GROUPS

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Let $G$ be an Abelian group of order $A > \aleph_0$. It has been shown [4, Theorem 9] that there exist $2^A$ subgroups of $G$ of order $A$, and that the intersection of all such subgroups is 0. In this paper, this result is improved to the following: If $\aleph_0 \leq B \leq A$ and $A > \aleph_0$, then an Abelian group of order $A$ has $2^A$ subgroups of index $B$, and the intersection of all such subgroups is 0. In addition, it is shown that there is a set of $2^A$ subgroups $H_\alpha$ of index $B$ such that $G/H_\alpha \cong G/H_\alpha'$ for all $\alpha, \alpha'$.

Baer [1, p. 124] showed that if $G$ is an Abelian $p$-group which is the direct sum of $A$ cyclic groups of bounded order, then $G$ has $2^A$ subgroups of index $p$ (here $A$ may equal $\aleph_0$). The proof in the present paper is accomplished by extending Baer's result in an obvious manner to a wider class of $p$-groups, and then reducing all other cases to this one.

We shall use $+$ and $\sum$ to denote direct sums, and $o(S)$ to denote the number of elements in $S$.

**Lemma.** Let $H \neq 0$ be an Abelian group, and let $G = \sum H_\alpha$, $\alpha \in S$, $H_\alpha \cong H$ for all $\alpha \in S$, $o(S) = A \geq \aleph_0$. Then there are at least $2^A$ subgroups $K_\beta$ of $G$ such that $G/K_\beta \cong H$.

**Proof.** (This proof is the same as Baer's, and is included only for the sake of completeness.) Identify $H_\alpha$ with $H$. Let $\epsilon_\alpha$ be 0 or 1 for each $\alpha, \alpha \in S$. Let $K$ be the set of elements of $G$ such that $h_{\alpha_0} = \sum \epsilon_\alpha h_\alpha$, $\alpha \neq \alpha_0$. Then it is easy to verify that $K$ is a subgroup of $G$ and $G = H_{\alpha_0} + K$. If $\epsilon_\alpha = 0$, then $H_\alpha \subseteq K$, but if $\epsilon_\alpha = 1$, then $H_\alpha \cap K = 0$. Thus all of the $K$'s are distinct, and the lemma is proved.

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Theorem. Let $G$ be an Abelian group of order $A > 1$, and let $N_0 \leq B \leq A$. Then

(i) there are exactly $2^A$ subgroups $H_a$ of index $B$ and order $A$,
(ii) the intersection of the subgroups in (i) is 0,
(iii) there exist $2^A$ subgroups $K_{i\beta}$ of index $B$ and order $A$, such that $G/K_{i\beta} \cong G/K_{j\beta} \cong \sum C_\gamma$ where either

(a) all $C_\gamma$ are cyclic of prime order, or
(b) all $C_\gamma$ are $p^\infty$ groups (not necessarily fixed).

Remarks. The condition that $o(H_a) = A$ or $o(K_{i\beta}) = A$ is automatically satisfied by subgroups of index $B$ unless $B = A$. If (ii) is true, then it is clear that there is a set of $A$ subgroups, each of index $B$ and order $A$, whose intersection is 0. Finally, since there are at most $2^A$ subgroups of $G$, (iii) implies (i), hence only (ii) and (iii) need be proved.

Proof. Case 1. $G = \sum C_\gamma$, $\gamma \in S$, $o(C_\gamma) = p$, $p$ a fixed prime. Then $o(S) = A$. By omitting $B$ and retaining $A$ summands, one obtains a subgroup $K$ of index $B$ and order $A$. By the lemma, $K$ has $2^A$ subgroups $K_{i\beta}$ of index $p$ in $K$. The $K_{i\beta}$ are thus of order $A$ and of index $B$ in $G$, and (iii) is satisfied. Since any given summand could have been omitted in obtaining $K$, (ii) is satisfied.

Case 2. $G = \sum C_\gamma$, $\gamma \in S$, where $C_\gamma$ is cyclic of order $p^\infty$. Again $o(S) = A$, and $G/pG$ is of the type considered in Case 1. Hence there are $2^A$ subgroups $K^*_{i\beta}$ of $G/pG$ as in (iii). Therefore there are $2^A$ subgroups $K_{i\beta}$ of $G$ satisfying (iii). To obtain (ii), first omit from $G$ one summand containing a nonzero component of a given nonzero element $g$. Since the group $G^*$ thus obtained has finite index in $G$, one may then proceed as above.

Case 3. $G = \sum C_\gamma$, $\gamma \in S$, where $C_\gamma$ is a $p^\infty$ group, $p$ fixed. The proof is nearly identical to that in Case 1.

Case 4. $G$ is a $p$-group. Then [3, Theorem 6] there exists a pure (= servant) subgroup $M$ of $G$ such that (a) $M = \sum C_\gamma$, $\gamma \in S$, where $C_\gamma$ is cyclic, and (b) $G/M = \sum D_\delta$, where the $D_\delta$ are $p^\infty$ groups. If $\gamma_1, \ldots, \gamma_n \in S$, then $C_{\gamma_1} + \cdots + C_{\gamma_n}$ is a pure subgroup of $M$ since it is a direct summand thereof, consequently $C_{\gamma_1} + \cdots + C_{\gamma_n}$ is a pure subgroup of $G$. But since it is also of bounded order, we have [2, Theorem 5] (c) $C_{\gamma_1} + \cdots + C_{\gamma_n}$ is a direct summand of $G$ for all $\gamma_1, \ldots, \gamma_n \in S$.

If $o(G/M) = A$, then by (b) and Case 3, (iii) is satisfied for $G/M$, hence also for $G$.

If $o(G/M) < A$, then $o(M) = A$, and if $V$ is a set of representatives of the cosets of $M$, then the subgroup $L$ generated by $V$ has order less than $A$, and $M \cup L = G$. The summands $C_\gamma$ containing any com-
ponent of any element of \( M \cap L \) are fewer than \( A \) in number. Hence there exists a subgroup \( N \) of \( M \) such that \( N \supseteq M \cap L \), \( o(M/N) = A \), and \( M/N \) is isomorphic to a direct sum of some of the \( C_\gamma \). Therefore by Case 2, (iii) is true for \( M/N \), and therefore there are \( 2^A \) subgroups \( K_\beta^* \) of \( M \) containing \( N \), and therefore \( M \cap L \), such that the factor groups \( M/K_\beta^* \) are as in (iii). It then follows from the isomorphism theorem that (iii) is satisfied for \( G \).

To prove (ii), note that by (\( \gamma \)), any element of \( M \) may be omitted by a subgroup of finite index in \( G \), hence from the above, by a subgroup \( H_a \) of index \( B \) and order \( A \). If \( o(G/M) > B \), then by (\( \beta \)) and Case 3 (perhaps for a cardinal smaller than \( A \)), any nonzero element of \( G/M \) may be omitted by a subgroup of order \( o(G/M) \) and of index \( B \) in \( G/M \), hence any element of \( G \) outside of \( M \) may be omitted by a subgroup of order \( o(G/M) \) and of index \( B \) in \( G \). The same is true if \( o(G/M) = B = A \). If \( o(G/M) < B \) or \( o(G/M) = B < A \), then any subgroup of \( G \) of order \( A \) and index \( B \) in \( M \) (omit \( B \) and keep \( A \) summands \( C_\gamma \) in (\( \alpha \))) has the property of omitting all elements outside \( M \) and of having the right order and index in \( G \).

Case 5. \( G \) is periodic. Then \( G = \sum G_p \), where \( G_p \) is the \( p \)-component of \( G \), and \( A = \sum o(G_p) \). Let

\[
\begin{align*}
S_1 &= \{ p \mid o(G_p) \leq N_0 \}, \\
S_2 &= \{ p \mid o(G_p) > N_0, \text{ and (iii) (a) holds for } G_p \}, \\
S_3 &= \{ p \mid o(G_p) > N_0, \text{ and (iii) (a) does not hold for } G_p \}, \\
N_i &= \sum_{p \in S_i} G_p, \quad i = 1, 2, 3.
\end{align*}
\]

Then either \( o(N_2) = A \) or \( o(N_3) = A \). Now if \( p \in S_2 \), then by Case 4, \( G_p \) has \( 2^{o(G_p)} \) subgroups \( K_{p, \gamma} \) of order \( o(G_p) \) and index \( \min (o(G_p), B) \) such that the \( G_p/K_{p, \gamma} \) are as in (iii)(a). Hence if \( K = N_1 + N_3 + \sum K_{p, \gamma}, p \in S_2 \), then \( G/K \) is isomorphic to a fixed product of type (a) for all choices of \( K \), and \( K \) is of order \( A \) and of index \( \min (o(N_2), B) \) in \( G \). There are

\[
\prod_{p \in S_2} 2^{o(G_p)} = 2^{\sum_{p \in S_1} o(G_p)} = 2^{o(N_2)}
\]

such \( K \). Similarly there are (at least) \( 2^{o(N_3)} \) \( K_\beta \) such that \( K_\beta \) is of order \( A \), of index \( \min (o(N_3), B) \) in \( G \), and such that \( G/K_\beta \) is of type (b). Hence (iii) is satisfied, since either \( o(N_2) \) or \( o(N_3) \) equals \( A \).

To prove (ii), let \( g \in G \) have component \( g_{p_0} \neq 0 \). If \( o(G_{p_0}) < B \), or \( o(G_{p_0}) = B < A \), then omit the summand \( G_{p_0} \) from \( G \), and find a subgroup of order \( A \) and of index \( B \) in the resulting \( G^* \). If \( o(G_{p_0}) > B \) or \( o(G_{p_0}) = B = A \), then by Case 4, there exists a subgroup \( K_{p_0} \) of \( G_{p_0} \)
of index $B$ in $G_{p_0}$ and, in case $B=A$, of order $A$, which omits $g_{p_0}$.  Then $K_{p_0} + \sum G_p$, $p \neq p_0$, omits $g$ and has the required properties.  Thus (ii) holds.

Case 6. $G$ is not periodic. Let $F = \{f_a\}$ be a maximal independent set of elements of $G$. Let $L_k$, $k=2, 3, \ldots$, be the subgroup generated by the maximal independent set of elements $\{k f_a\}$. If $o(F) < A$, then $o(L_k) < A$ and $o(G/L_k) = A$. If $o(F) = A$, then the $f_a$ lie in distinct cosets of $L_k$, so again $o(G/L_k) = A$. But $G/L_k$ is periodic, and therefore by Case 5, (iii) is true, and in (ii), $\cap H_a \subseteq L_k$. Since $\cap L_k = 0$, $k=2, 3, \ldots$, (ii) also holds.

**Bibliography**


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