CRITICAL POINTS OF RATIONAL FUNCTIONS WITH SELF-INVERSIVE POLYNOMIAL FACTORS

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1. **Introduction.** A polynomial is said to be *self-inversive* if its zeros are symmetric in the unit circle $C: |z| = 1$. Let E be an arbitrary subset of the finite complex plane Z, and let $\mathfrak{N}(f, E)$ and $\mathfrak{P}(f, E)$ denote respectively the total multiplicity of the zeros and poles in E of a function f. Let $\mathfrak{Q}(f, E)$ denote the number of distinct poles of f in E, and let f' denote the derivative of f. In this notation, Cohn's Theorem[1] states that, if f is a self-inversive polynomial, then

$$\mathfrak{N}(f', |z| > 1) = \mathfrak{N}(f, |z| > 1).$$

The theorem of Lucas [3, p. 14] states that if g is any polynomial for which $\mathfrak{N}(g, |z| > 1) = 0$, then $\mathfrak{N}(g', |z| > 1) = 0$. A result (Bôcher's Theorem)[2] due to Walsh states, in effect, that if ϕ is a rational function for which $\mathfrak{N}(\phi, |z| > 1) = \mathfrak{P}(\phi, |z| \leqq 1) = 0$, then

$$\mathfrak{N}(\phi', |z| \leqq 1) = \mathfrak{N}(\phi, |z| \leqq 1) - 1,$$

provided $\phi = k/K$ with degree $K \leqq$ degree k.

These three theorems are special cases of the following, which is our principal result.

THEOREM 1. *Let $\phi = k/K$ be a rational function in which the degree of the polynomial k is greater than that of the polynomial K. Let $k = fg$ and $K = FG$ where f, g, F, G are polynomials, f and F are self-inversive, and $\mathfrak{N}(g, |z| > 1) = \mathfrak{N}(G, |z| < 1) = 0$. Then*

$$\mathfrak{N}(\phi', |z| > 1) = \mathfrak{N}(\phi, |z| > 1) + \mathfrak{Q}(\phi, |z| \geqq 1). \tag{1.1}$$

COROLLARY 1.

$$\mathfrak{N}(\phi', |z| \leqq 1) = \mathfrak{N}(\phi, |z| \leqq 1) + \mathfrak{Q}(\phi, |z| < 1) - 1.$$

The proof of Theorem 1, given in §3, is simple in principle though slightly complicated in detail owing to the possible presence of poles on C. Lemma 1 shows that the only zeros of ϕ' on C are the multiple zeros there of ϕ. We may therefore vary ϕ continuously without

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[1] See [2]. A simpler proof of this theorem was given in [1]. Another, [6], was published since the announcement of the present paper.

[2] See [4, pp. 97–99]. This book gives a number of other interesting results on the zeros of self-inversive polynomials, particularly on pp. 52–55, 132–135, 159–163.

changing $\mathfrak{N}(\phi', |z|<1)-\mathfrak{P}(\phi', |z|<1)$ until we arrive at a rational function for which this number may be counted. If applied to the case when $K\equiv g\equiv 1$, this method would yield a new and simpler proof of Cohn's theorem.

2. Three lemmas.

LEMMA 1. *Under the hypotheses of Theorem* 1, $\phi'(t)=0$ *for* t *on* C *if and only if* t *is a multiple zero of* $\phi(z)$.

PROOF. Without loss of generality, we may assume that $t=1$. If $\phi(1)\neq 0$, then $\phi'(1)=0$ would imply

$$\begin{aligned}(2.1)\qquad 0 = \phi'(1)/\phi(1) &= [f'(1)/f(1)] + [g'(1)/g(1)] \\ &\quad - [F'(1)/F(1)] - [G'(1)/G(1)].\end{aligned}$$

Let the zeros of f, g, F, G, be denoted by a_j, b_j, A_j, B_j, respectively, and for any complex number z let z^* denote $(1-z)^{-1}$. Then u, v, U, V, defined by

$$mu = f'(1)/f(1) = \sum_1^m a_j^*, \qquad MU = F'(1)/F(1) = \sum_1^M A_j^*,$$

$$nv = g'(1)/g(1) = \sum_1^n b_j^*, \qquad NV = G'(1)/G(1) = \sum_1^N B_j^*,$$

where m, n, M, N denote the degrees of f, g, F, G, are the centroids of the a_j^*, b_j^*, A_j^*, and B_j^* respectively. Since $w=(1-z)^{-1}$ maps the closed interior of C upon the half-plane Re $(w)\geqq 1/2$ and preserves symmetry in C, we have

$$\text{Re}\,(u) = \text{Re}\,(U) = 1/2, \qquad \text{Re}\,(v) \geqq 1/2, \qquad \text{Re}\,(V) \leqq 1/2;$$

$$\begin{aligned}\text{Re}\,[\phi'(1)/\phi(1)] &= \text{Re}\,[mu + nv - MU - NV] \\ &\geqq (m+n-M-N)/2 > 0.\end{aligned}$$

As this contradicts (2.1), $\phi'(1)=0$ if and only if also $\phi(1)=0$.

The following generalization of Rouché's Theorem may be proved by a method similar to that given in [**5**, p. 191–192] for the usual Rouché Theorem. It will be convenient to write $(\mathfrak{N}-\mathfrak{P})(f, E)$ in place of $\mathfrak{N}(f, E)-\mathfrak{P}(f, E)$.

LEMMA 2. *Let* D *be the interior domain of a simple closed rectifiable curve* κ. *Let* f *and* g *be regular in a domain containing the closure of* D *except perhaps for poles in* D. *If* $|g(z)|<|f(z)|$ *on* κ, *then*

$$(\mathfrak{N}-\mathfrak{P})(f+g, D) = (\mathfrak{N}-\mathfrak{P})(f, D).$$

PROOF. Let $I(\lambda)=(2\pi i)^{-1}\int_\kappa[f'(z)+\lambda g'(z)][f(z)+\lambda g(z)]^{-1}dz$. Then $I(\lambda)$ is a continuous function of λ for $0\leqq\lambda\leqq 1$ and, since it takes only integer values, we conclude that $I(1)=I(0)$.

LEMMA 3. *With D, κ as in Lemma 2, let $\psi_\alpha(z)$ be a meromorphic function of z in D with no zeros on κ for each α in $A: 0\leqq\alpha\leqq 1$. If $\psi_\alpha(z)$ is a continuous function of (α, z) for $z\in\kappa$, $\alpha\in A$, then*

$$(\mathfrak{N}-\mathfrak{P})(\psi_1, D) = (\mathfrak{N}-\mathfrak{P})(\psi_0, D).$$

PROOF. Use Lemma 2 and a simple covering argument with respect to A.

3. **Proof of Theorem 1.** If ϕ has no zeros and no poles on C, set

$$\phi(z) = \sigma\prod(z-\alpha_i)\prod(z-\beta_i)\prod(z-\gamma_i)^{-1}\prod(z-\delta_i)^{-1}$$

where $|\alpha_i|<1$, $|\beta_i|>1$, $|\gamma_i|<1$, $|\delta_i|>1$, $\sigma=$const. Let

$$\phi_\rho(z) = \sigma\prod(z-\rho\alpha_i)\prod(\rho z-\beta_i)\prod(z-\rho\gamma_i)^{-1}\prod(\rho z-\delta_i)^{-1}.$$

Lemma 1 shows that ϕ_ρ' has no zeros on C for $0\leqq\rho\leqq 1$. Since, evidently, ϕ_ρ' is a continuous function of (z, ρ) for $|z|=1$, $0\leqq\rho\leqq 1$, we may apply Lemma 3 and conclude that

$$(\mathfrak{N}-\mathfrak{P})(\phi_1', |z|<1) = (\mathfrak{N}-\mathfrak{P})(\phi_0', |z|<1).$$

However, $\phi_0(z)=\sigma z^t$, where $t=(\mathfrak{N}-\mathfrak{P})(\phi, |z|<1)$. It follows that

$$(3.1)\qquad (\mathfrak{N}-\mathfrak{P})(\phi', |z|<1) = (\mathfrak{N}-\mathfrak{P})(\phi, |z|<1) - 1,$$

which yields (1.1) due to the relations

$$\mathfrak{P}(\phi', E) = \mathfrak{P}(\phi, E) + \mathfrak{Q}(\phi, E),$$

$$Z: |z|<\infty, \mathfrak{N}(\phi', Z) = \mathfrak{N}(\phi, Z) + \mathfrak{Q}(\phi, Z) - 1.$$

If $\phi=0$ or ∞ on C, we may write $k=h_1h_2$, $K=H_1H_2$, where

$$\mathfrak{N}(h_1, |z|\neq 1) = \mathfrak{N}(H_1, |z|\neq 1) = \mathfrak{N}(h_2, |z|=1)$$
$$= \mathfrak{N}(H_2, |z|=1) = 0,$$

and apply (3.1) and Lemma 3 to

$$\Phi(\rho) = h_1(\rho z)h_2(z)/H_1(z/\rho)H_2(z), \qquad \rho>1.$$

We may complete the proof by choosing ρ sufficiently near 1.

COROLLARY 2. *With ϕ defined as in Theorem 1, let $\psi=1/\phi$. Then,* $\mathfrak{N}(\psi', |z|>1)=\mathfrak{N}(\psi, |z|>1)+\mathfrak{Q}(1/\psi, |z|=1)+\mathfrak{Q}(\psi, |z|>1)$.

COROLLARY 3. $\mathfrak{N}(\phi', |z|>1)$ *and* $\mathfrak{N}(\psi', |z|>1)$ *are left unaltered if ϕ is multiplied by a polynomial all of whose zeros are within C.*

References

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