NOTE ON THE HEAT EQUATION[1]

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If $u(x, t)$ is the temperature of an infinite insulated rod, then

$$H(a, b; t) = \int_a^b u(x, t)dx \tag{1}$$

is the amount of heat at time t in the interval $[a, b]$. We prove an existence and uniqueness theorem for the case in which the heat, rather than the temperature, is prescribed initially. If the prescribed heat is not absolutely continuous, its derivative $\partial H/\partial b$, the initial temperature, may not determine H, so that we are dealing with a more general problem than the usual one.

The following lemma, which we did not find in the literature, will be useful.

LEMMA 1. *Let $f(x)$ be measurable, and $|f(x)| \leqq Me^{cx^2}$. Define*

$$W_r(x, t) = (4\pi t)^{-1/2} \frac{\partial^r}{\partial x^r} \left[e^{-x^2/4t}\right]. \tag{2}$$

Then, for every integer $r \geqq 0$, the integral

$$u_r(x, t) = \int_{-\infty}^{\infty} W_r(x - \xi, t)f(\xi)d\xi \tag{3}$$

is abolutely convergent and satisfies

(a) $\qquad \partial u_r/\partial x = u_{r+1}, \qquad \partial u_r/\partial t = u_{r+2} \qquad$ *on* $0 < t < 1/4c$,

and, for all t_0 satisfying $0 < t_0 < 1/4c$,

(b) $\qquad |u_r(x, t)| \leqq Kt^{-r/2}e^{Nx^2}, \qquad$ *on* $0 < t \leqq t_0$,

where $K = K(r, t_0)$, and $N = N(t_0)$.

PROOF. It is well known that $W_r(x, t)$ is analytic, satisfies the heat equation, and has the specific form

$$W_r(x, t) = \frac{1}{(2\pi)^{1/2}} (2t)^{-(r+1)/2} H_r\left(\frac{x}{(2t)^{1/2}}\right) e^{-x^2/4t}, \tag{4}$$

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where $H_r(z)$ is the Hermite polynomial of degree r in $z=x/(2t)^{1/2}$. The absolute convergence of (3) on $0<t<1/4c$ is obvious from (4) and $|f(x)|\leqq Me^{cx^2}$.

To prove the first equation of (a), note

$$(5)\quad h^{-1}[u_r(x+h,t)-u_r(x,t)]=\int_{-\infty}^{\infty}W_{r+1}(x+\theta h-\xi,t)f(\xi)d\xi,$$

$$0\leqq\theta\leqq 1,$$

by the Law of the Mean. But, as $|\xi|\to\infty$,

$$|W_{r+1}(x+\theta h-\xi,t)|=O(|x-\xi|+1]^{r+1}e^{-(|x-\xi|+1)^2/4t}),$$

and is bounded in finite intervals. Hence, since $t<1/4c$, the integrand in (5) tends *dominatedly* [3, p. 168] to $W_{r+1}(x-\xi,t)f(\xi)$ as $h\to 0$, which shows that

$$\lim_{h\to 0}h^{-1}[u_r(x+h,t)-u_r(x,t)]=\int_{-\infty}^{\infty}W_{r+1}(x-\xi,t)f(\xi)d\xi,$$

proving the first equation of (a). A similar proof applies to the second equation.

To prove (b), note that the right side of (3) is by (4) bounded by a sum of terms of the form

$$Mt^{-s}\int_{-\infty}^{\infty}|x-\xi|^{s}e^{-((x-\xi)^2/4t)+c\xi^2}d\xi\qquad [s\leqq r].$$

Completing the square, the exponent becomes

$$[(1-4ct)/4t][\xi-x/(1-4ct)]^2+cx^2/(1-4ct).$$

Writing $\eta=t^{-1/2}[\xi-x/(1-rct)]$, and noting that $c/(1-4ct)<c/(1-4ct_0)=N'(t_0)$, we bound the right side of (3) by a sum of terms

$$M'e^{N'x^2}t^{-s}\int_{-\infty}^{\infty}|\eta t+N'xt|^{s}e^{-(1-4ct)\eta^2}d\eta.$$

Since $1-4ct\geqq 1-4ct_0=L>0$, another bound is

$$M'e^{N'x^2}t^{-s/2}\int_{-\infty}^{\infty}|\eta+N'xt^{1/2}|^{s}e^{-L\eta}\,d\eta.$$

For any $N>N'(t_0)$ and suitable $K=K(t_0,r)$, this evidently implies (b).

COROLLARY. *In Lemma* 1, $u_r\in C^\infty$ *and* $\partial^{m+n}u_r/\partial x^m\partial t^n=u_{r+m+2n}$.

We shall also use the following essentially known facts.

THEOREM A. *Let $f(x)$ be measurable, and let* $|f(x)| \leqq Me^{cx^2}$, $c>0$. *Then*

$$u(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t) f(\xi) d\xi, \tag{6}$$

is absolutely convergent for $0<t<1/4c$, *and*

(α) $u(x, t) \in C^\infty$ *and* $u_{xx} = u_t$, $\qquad 0 < t < 1/4c$,

(β) $|u(x, t)| \leqq Ke^{Nx^2}$, $0 < t \leqq t_0 < 1/4c$, $K = K(t_0)$, $N = N(t_0)$.

If $f(x)$ has bounded variation on every finite interval, and $2f(x) = f(x^+) + f(x^-)$, *then*

$$(\gamma) \qquad \lim_{t\to 0^+} u(x, t) = f(x).$$

If $f(x)$ is continuous, then

$$(\gamma') \qquad \lim_{t\to 0^+, x\to x_0} u(x, t) = f(x_0),$$

so that if $u(x, 0)$ is defined as $f(x)$, then $u(x, t)$ is continuous for $0 \leqq t < 1/4c$.

PROOF. For (α), (γ), and (γ') see [1, p. 298]; (β) is part of Lemma 1.

THEOREM B. *Conversely, if $u(x, t)$ is defined for $0<t\leqq t_0$ and satisfies*

(α) $u(x, t) \in C^2$ *and* $u_{xx} = u_t$, $\qquad 0 < t \leqq t_0$,

(β) $|u(x, t)| \leqq Ke^{Nx^2}$, $0 < t \leqq t_0$, *for some finite constants*

$$K = K(t_0), \qquad N = N(t_0),$$

(γ) $\lim_{t\to 0^+} u(x, t) | = 0$, *for all x*,

then

$$u(x, t) \equiv 0, \ 0<t\leqq t_0.$$

The usual theorem, assuming simultaneous continuity in x and t, $t \geqq 0$, is in [2, p. 88]. To prove our slightly sharper result, note that by the usual theorem, if $0<\tau<t<1/4N$, then

$$u(x, t) = \lim_{\tau\to 0^+} \int_{-\infty}^{\infty} u(\xi, \tau) W_0(x - \xi, t - \tau) d\xi.$$

If $\tau<t/2$, then $W_0(x-\xi, t-\tau) \leqq (2\pi t)^{-1/2} e^{-(x-\xi)^2/2t}$. Hence ($\beta$) gives dominated convergence [3, p. 168], and we can pass to the limit

under the integral sign. We conclude that $u(x, t)=0$ for all t such that $t<1/4N$. Suppose t_1 is the l.u.b. of the values of t for which $\tau<t \to u(x, \tau)=0$, and suppose $t_1<t_0$. Then, by continuity, $u(x, t_1)=0$, and we may repeat the preceding argument to prove that $u(x, t)\equiv 0$, $t\leqq t_1+\eta$, for some positive η. This contradiction shows that $t_1=t_0$, completing the proof of Theorem 2.

We shall now prove our existence theorem.

THEOREM 1. *Let $H(x)$ have bounded variation on every finite interval, let $2H(x)=H(x^+)+H(x^-)$, $H(0)=0$, and let $|H(x)|\leqq Me^{cx^2}$. Then the improper integral*

$$(7)\quad u(x, t) = \int_{-\infty}^{\infty} W_0(x-\xi, t)dH(\xi) = \lim_{A\to\infty}\int_{-A}^{A} W_0(x-\xi, t)dH(\xi)$$

exists for $0<t<1/4c$, and satisfies

(a) $u(x, t) \in C^\infty$ *and* $u_{xx} = u_t$ *on* $0 < t < 1/4c$,

(b) $|u(x, t)| \leqq Kt^{-1/2}e^{Nx^2}$ *on* $0 < t \leqq t_0 < 1/4c$,

(c) $\lim_{t\to 0^+}\int_0^x u(\xi, t)d\xi = H(x)$ *for all* x,

(d) $\int_0^x u(\xi, t)d\xi \leqq Le^{Nx^2}$ *on* $0 < t \leqq t_0 < 1/4c$,

where $L=L(t_0)$ and $N=N(t_0)$.

PROOF. If $0<t<1/4c$, then by direct calculation,

$$u(x, t) = \lim_{A\to\infty}\int_{-A}^{A} W_0(x-\xi, t)dH(\xi)$$
$$= \lim_{B\to\infty}\left\{\left[W_0(x-\xi, t)H(\xi)\right]_{-A}^{A} + \int_{-A}^{A} W_1(x-\xi, t)H(\xi)d\xi\right\}.$$

By (4) and $|H(x)|\leqq Me^{cx^2}$, the term in square brackets tends to zero. Hence

$$(8)\qquad u(x, t) = \int_{-\infty}^{\infty} W_1(x-\xi, t)H(\xi)d\xi.$$

Lemma 1 assures us that (8) converges absolutely for $0<t<1/4c$, and has the properties (a), (b). Hence we can interpret $u(x, t)$ as coming from an initial dipole distribution of density $H(x)$.

To prove (c), integrate (8). Defining

$$H(x, t) = \int_0^x u(\xi, t) d\xi, \tag{9}$$

we get

$$H(x, t) = \int_{-\infty}^{\infty} [W_0(x - \xi, t) - W_0(-\xi, t)] H(\xi) d\xi, \tag{9'}$$

where the Fubini theorem is used to interchange the order of integration. Letting t tend to zero, and using (γ) of Theorem A, we get (c). To prove (d), we apply (β) of Theorem A to both terms of (9′) and add.

To prove uniqueness, we shall want the following result.

THEOREM C. *If $u(x, t) \in C^2$, $u_{xx} = u_t$, and $|H(x, t)| \leqq Ke^{Nx^2}$, where $H(x, t)$ is defined by* (9) *for all $0 \leqq t \leqq T$, then*

$$u(x, t) = \int_{-\infty}^{\infty} W_1(x - \xi, t) H(\xi, 0) d\xi, \qquad \textit{for } t < 1/4N. \tag{10}$$

PROOF. We first note that

$$H_t = \int_0^x u_t(\xi, t) d\xi = \int_0^x u_{xx}(\xi, t) d\xi = u_x(x, t) - u_x(0, t),$$

$$H_{xx} = u_x(x, t).$$

Hence, if we define $\overline{H}(x, t) = H(x, t) + \int_0^t u_x(0, \tau) d\tau$, then $\overline{H}(x, t)$ satisfies the heat equation, $\overline{H}_{xx} = \overline{H}_t$. Also $\overline{H}(x, t) \in C^2$ and $|\overline{H}(x, t)| \leqq K' e^{Nx^2}$. Hence Theorem B applies, and

$$\overline{H}(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t) \overline{H}(\xi, 0) d\xi = \int_{-\infty}^{\infty} W_0(x - \xi, t) H(\xi, 0) d\xi.$$

Using part (a) of Lemma 1, we differentiate with respect to x and obtain (10), q.e.d.

THEOREM 2. *Let $u(x, t) \in C^2$, $u_t = u_{xx}$, $|H(x, t)| \leqq Ke^{Nx^2}$, where $H(x, t)$ is defined by* (9), *for all $0 < t \leqq T$, and $H(x, t) \to 0$ as $t \to 0^+$ for almost all x. Then $u(x, t) \equiv 0$ on $0 < t \leqq T$.*

PROOF. Let $0 < \tau < t < \min [1/4N, T]$, and $u_1(x, t - \tau) = u(x, t)$, $H_1(x, t - \tau) = H(x, t)$. Then for $t \geqq \tau$, $u_1(x, t - \tau)$, $H_1(x, t - \tau)$ satisfy the conditions of Theorem C. Hence

$$u_1(x, t - \tau) = \int_{-\infty}^{\infty} W_x(x - \xi, t - \tau) H_1(\xi, 0) d\xi,$$

or

$$u(x, t) = \int_{-\infty}^{\infty} W_x(x - \xi, t - \tau)H(\xi, \tau)d\xi.$$

By hypothesis $H(\xi, \tau) \to 0$ almost everywhere *dominatedly* as $\tau \to 0$. Hence, using the Lebesgue dominated convergence theorem again, we have $u(x, t) \equiv 0$, $0 < t < \min [1/4N, T]$. If $1/4N < T$, we repeat the argument, considering $\bar{u}(x, t) = h(x, t + 1/4N)$ instead of $u(x, t)$: $\bar{u}(x, 0) = 0$ by the continuity of $u(x, t)$ in $0 < t < T$. This shows $u(x, t) \equiv 0$, $0 < t < \min [1/4N, T]$. The proof is complete after at most $[4NT]$ steps.[2]

COROLLARY. *Under the hypotheses of Theorem* 1, $u(x, t)$ *as defined by* (6) *is the only function satisfying conditions* (a), (c), (d).

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3. E. J. McShane, *Integration*, Princeton University Press, 1944.

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[2] J. L. B. Cooper (J. London Math. Soc. vol. 25 (1950) pp. 173–180) has proved a result related to Theorem 2. However, he requires that $\int_a^b g(x)u(x, t)dx \to 0$ for all bounded integrable $g(x)$; we require only $\int_a^b u(x, t)dx \to 0$, for all a, b. Cf. also J. Kampé de Fériet, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 1527–1929.