THE ALGEBRA OF BOUNDED FUNCTIONS

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1. **Introduction.** The Banach algebra $B(X)$ of *all* bounded complex-valued functions on a set X has been considered by several writers (cf., e.g. [**10**]). It is our purpose to give an ideal-theoretic characterization of this algebra. The main result (Theorem 3) states necessary and sufficient conditions that a B^*-algebra be equivalent to a $B(X)$ on an essentially unique set X.

Our method consists in establishing a lattice anti-isomorphism between the lattice of subsets of X and the lattice of all those ideals of $B(X)$ which are defined by annihilation.

These results are also applied to the study of certain p-rings.

2. By a B^*-algebra is meant a Banach algebra over the complex numbers such that, to each element x, there corresponds a unique element x^* (the adjoint of x), with the following properties: (i) $(x^*)^*=x$, (ii) $(xy)^*=y^*x^*$, (iii) if α, β are complex numbers, and $\bar{\alpha}$, $\bar{\beta}$ are their complex conjugates, then $(\alpha x+\beta y)^*=\bar{\alpha}x^*+\bar{\beta}y^*$, (iv) $\|x^*x\|=\|x\|^2$.

Let X be any set and denote by $B(X)$ the set of all bounded complex-valued functions defined on X. For $f(x)\in B(X)$ define $\|f(x)\| =\sup_{x\in X}|f(x)|$. Under the usual definitions of addition and multiplication and with the norm defined above, and $[f(x)]^*=\overline{f(x)}$, $B(X)$ is a commutative B^*-algebra with identity.

If G is any subset of an arbitrary commutative ring K, denote by $R(G)$ the set of elements k in K which satisfy $Gk=0$. $R(G)$ is an ideal of K which we call an *annulet*. It is closed if K is a topological ring.

$E(X)$ will denote an arbitrary algebra of bounded functions on the set X, while the algebra of all functions which are zero at all but a finite set of points of X is denoted by $B_0(X)$.

If S is any subset of X, and G a subset of $E(X)$, let $A(S)$ be the set of all $f\in E(X)$ which satisfy $f(S)=0$, and $N(G)$ the set of all $x\in X$ for which $G(x)=0$. It is clear that $A(S)$ is a closed ideal of $E(X)$.

If S, T are any subsets of X, the logical sum and product are denoted by $S+T$ and $S\cap T$ respectively, while $X-S$ denotes the set of those elements in X which are not in S.

LEMMA 1. *If $E(X)$ is an algebra of bounded functions containing the*

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algebra $B_0(X)$, *then*

(1) $R[A(S)]=A(X-S)$ *for all* $S\leqq X$,

(2) $N[A(S)]=S$ *for all* $S\leqq X$,

(3) $R(G)=A[X-N(G)]$ *for every subset* G *of* E.

PROOF. (1) $A(S)A(X-S)=0$ implies that $A(X-S)\leqq R[A(S)]$. If $S=X$, then $A(S)=0$ so that $R[A(S)]=E$. If we accept the convention that all functions vanish on the empty set, we see that (1) holds for $S=X$. Assume $S<X$ and let $f\in R[A(S)]$ so that $A(S)f=0$. Let $x\in X-S$. Then there exists $g\in E$ satisfying $g(S)=0$ and $g(x)=1$. Thus $g\in A(S)$ but since $gf=0$ we must have $f(x)=0$. That is, $f(X-S)=0$ or $f\in A(X-S)$ proving (1).

(2) Since $A(S)[S]=0$ we have $S\leqq N[A(S)]$. To complete the proof we must show $N[A(S)]\leqq S$. We shall assume $x\notin S$ and show $x\notin N[A(S)]$. Write $X=x+S+U$ and define f by

$$f(S+U)=0 \quad \text{and} \quad f(x)=1.$$

Such an $f\in E(X)$ by assumption. Then $f\in A(S)$. But $f(x)=1$ implies $A(S)(x)\neq 0$ so that $x\notin N[A(S)]$ which shows (2).

(3) Let $f\in R(G)$ so that $Gf=0$. Let $x\in[X-N(G)]$ so that $x\notin N(G)$. This means there exists a function $g\in G$ such that $g(x)\neq 0$. But since $g(x)f(x)=0$ this implies $f(x)=0$. Hence $f[X-N(G)]=0$ and $f\in A[X-N(G)]$. The reverse implication follows by reversing the steps.

THEOREM 1. *Let* $E(X)$ *be an algebra of bounded functions which contains* $B_0(X)$. *Then the mapping* $S\to A(S)$ *is a duality (lattice anti-isomorphism) between the lattice of all subsets* S *of* X *and the lattice of all annulets of* $E(X)$.

PROOF. The mapping $S\to A(S)$ is one-one by (2) of Lemma 1. Clearly $S_1\geqq S_2$ implies $A(S_1)\leqq A(S_2)$ and the mapping is a duality. Every annulet has the form $A(S)$ by (3) of Lemma 1, and every $A(S)$ is an annulet by (1) of Lemma 1.

THEOREM 2. *In* $B(X)$ *the sum of two annulets is an annulet.*

PROOF. Denote by (a) the ideal generated by the element a. An ideal of $B(X)$ is an annulet if and only if it is a principal ideal generated by an idempotent. For, if e is idempotent $(e)=R(1-e)$, and if G is an annulet, $G=(e)$ for idempotent e by Theorem 2.3 of [3]. It is clear that if $G=A(S)$, the idempotent e is just the characteristic function of $X-S$. Now, let the given annulets be $A(S)$, $A(T)$ where $A(S)=(e)$, $A(T)=f$ for idempotents e and f which satisfy

$$e(S) = 0, \qquad f(T) = 0,$$
$$e(X - S) = 1, \qquad f(X - T) = 1.$$

Since $A(S)$ and $A(T)$ are contained in $A(S\cap T)$ it follows that $A(S)+A(T)\leqq A(S\cap T)$. It remains to show that $A(S\cap T)\leqq A(S)+A(T)$. Let $A(S\cap T)=(g)$ where $g(S\cap T)=0$, $g\{X-[S\cap T]\}=1$. Write $X=[S\cap T]+[S-(S\cap T)]+[T-(S\cap T)]+[X-(S+T)]$, a decomposition of X into four mutually disjoint sets. Then $g=e+f-ef$ on these four parts separately and hence $g=e+f-ef$. Hence $g\leqq A(S)+A(T)$ and $(g)\leqq A(S)+A(T)$, which completes the proof.

THEOREM 3. *A commutative B^*-algebra K containing an identity is isomorphic (in a norm and $*$ preserving manner) to an algebra $B(X)$ of all bounded complex-valued functions on an essentially unique set X if and only if*:

(1) *Every nonzero closed ideal of K contains a minimal ideal.*

(2) *The sum of two annulets is an annulet.*

PROOF. The minimal ideals of K are of the form e_iK for e_i a projection (self-adjoint idempotent) [**2**, Lemma 4.2]. Each e_iK is a field and by the Gelfand-Mazur theorem is isomorphic and isometric to the complex field. Consider the mapping $x\rightarrow\{e_ix\}$ for $x\in K$. Since each e_i is of norm 1, if we define $*$ and norm in the obvious way we have a $*$ homomorphism of K into the algebra of all bounded sequences. Since $\prod_i R(e_iK)$ is an intersection of maximal and hence [**1**, p. 8] closed ideals, it follows from (1) and a theorem of McCoy [**5**, p. 873], that K is isomorphic to a special subdirect sum of the e_iK. That is, K is isomorphic to an algebra $E(X)$ of bounded functions defined on the set X of minimal ideals e_iK, where if $k\in K$, by $k(e_iK)$ is meant the complex number λ defined by $e_ik=\lambda e_i$. Moreover, to each $x\in X$ and complex number λ, there exists $f\in E(X)$ such that $f(x)=\lambda$, and $f(X-x)=0$. Since $E(X)$ is a ring we have $E(X)\geqq B_0(X)$.

We wish to show that $E(X)$ contains all characteristic functions of subsets of X. For any subset $S\leqq X$, consider the annulets $A(S)$, $A(X-S)$. By Theorem 1,

$$A(S) \cup A(X - S) = A(0) = E(X),$$
$$A(S) \cap A(X - S) = A(X) = 0,$$

where $\cup$ indicates the lattice join of annulets. But, by (2),

$$A(S)\cup A(X-S)=A(S)+A(X-S).$$

Hence $E(X)=A(S)\oplus A(X-S)$ in the sense of direct sum of ideals. By a theorem of von Neumann [7, p. 7] it follows, since $E(X)$ contains an identity, that there exists an idempotent $e\leqq E(X)$ such that

$$A(S) = (1 - e)E, \qquad A(X - S) = eE.$$

Since $e\leqq A(X-S)$ this implies $e(X-S)=0$. $1-e\in A(S)$ implies $(1-e)(S)=0$ or $e(S)=1$. Hence $E(X)$ contains all characteristic functions and is therefore clearly dense in $B(X)$. Applying Theorem 6.4 of [2] the proof of sufficiency is complete.

By Theorem 1, if K were isomorphic to $B(X_1)$ and $B(X_2)$ there would exist a lattice isomorphism of the lattice of all subsets of X_1 upon the lattice of all subsets of X_2. In this sense the set X is unique.

The algebra $B(X)$ is certainly a B^*-algebra with identity, and satisfies (2) by Theorem 2. Every nonzero ideal of $B(X)$ (not only the closed ones) contains a minimal ideal, as in Theorem 16 of [5] mentioned previously. This completes the proof.

REMARK. The set X may also be considered to be the set of all maximal annulets of $B(X)$ (those maximal ideals which are annihilators of minimal ideals). Since $B(X)$ may be represented as the algebra $C(M)$ of all continuous functions on the compact space M of its maximal ideals, it is seen that M is the Stone-Čech compactification [9, p. 463] of the discrete space X of maximal annulets. An alternative proof of Theorem 3 could start with the representation $C(M)$.

3. The preceding results with slight modification apply to rings of functions from a discrete set X to a finite field.

In [6] it was shown that a ring of prime characteristic p, all of whose elements x satisfy $x^p=x$ (called a p-ring), is isomorphic to a subdirect sum of fields $GF(p)$. The ring $P(X)$ of all functions from a set X to $GF(p)$ (the complete direct sum of the fields) is generated by its idempotents, since an arbitrary function in $P(X)$ assumes only a finite number of distinct values and each is an integral multiple of the identity of the field $GF(p)$. Hence we have:

A p-ring K containing an identity element is isomorphic to a $P(X)$ if and only if:

(1) *Every annulet of K contains a minimal ideal.*

(2) *The sum of two annulets is an annulet.*

Specializing this to Boolean rings ($p=2$) we get:

A Boolean ring K containing an identity is isomorphic to the ring of all subsets of a set S if and only if it satisfies conditions (1) *and* (2) *above.*

This result is similar to one obtained by Stone [**8**, p. 98, Theorem 62].

The complete direct sum of the fields $GF(p^n)$ is generated by its idempotents when considered as an algebra over $GF(p^n)$ (although not as a ring). Hence we have:

Let K be an algebra (containing an identity element) over the Galois field $GF(p^n)$, all of whose elements satisfy the equation $x^{p^n}=x$. Then K is isomorphic to the algebra $W(X)$ of all functions from a set X to the field $GF(p^n)$ if and only if K satisfies the conditions (1) *and* (2) (*cf.* [**4**, *p.* 379, *Theorem* 3]).

References

1. I. Gelfand, *Normierte Ringe*, Mat. Sbornik N.S. vol. 9 (1941) pp. 3–24.

2. Irving Kaplansky, *Normed algebras*, Duke Math. J. vol. 16 (1949) pp. 399–418.

3. ———, *Projections in Banach algebras*, Ann. of Math. vol. 53 (1951) pp. 235–249.

4. N. H. McCoy, *Subrings of direct sums*, Amer. J. Math. vol. 60 (1938) pp. 374–382.

5. ———, *Subdirect sums of rings*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 856–877.

6. N. H. McCoy and D. Montgomery, *A representation of generalized Boolean rings*, Duke Math. J. vol. 3 (1937) pp. 455–459.

7. J. von Neumann, *Continuous geometry*, vol. 2, Princeton Lectures, 1936–37.

8. M. H. Stone, *The theory of representations for Boolean algebras*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 37–111.

9. ———, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481.

10. Bertram Yood, *Banach algebras of bounded functions*, Duke Math. J. vol. 16 (1949) pp. 151–163.

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