SOME THEOREMS ON n-HOMOGENEOUS CONTINUA

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1. **Introduction and definitions.** Bing [2] has given an example of a homogeneous indecomposable continuum in the plane. His example is homeomorphic with an example given by Moise [16] to show the existence of a nondegenerate plane continuum which is not an arc and is homeomorphic with each of its nondegenerate subcontinua. Whyburn [19, Theorem 3] has shown that such a continuum does not separate the plane. Jones [11, Theorem 2] has shown that every homogeneous bounded plane continuum which does not separate the plane is indecomposable. The purpose of this paper is to give some conditions under which a homogeneous continuum is indecomposable and to show that every bounded plane continuum possessing a certain type of homogeneity is a simple closed curve.

If M is a continuum and there exist n continua whose sum is M such that no one of them is a subset of the sum of the others, then M is said to be the *finished sum* of these n continua.

If the continuum M is the finished sum of n continua and is not the finished sum of $n+1$ continua, then M is said to be *indecomposable*[1] *under index n*.

A point set M is said to be *n-homogeneous* if for any n points $x_1, x_2, \cdots, x_n$ of M and any n points $y_1, y_2, \cdots, y_n$ of M there is a homeomorphism T of M onto itself such that $T(x_1+x_2+\cdots+x_n)=y_1+y_2+\cdots+y_n$. If in addition to these requirements, $T(x_i)=y_i$ $(1\leqq i\leqq n)$, then M is said to be *strongly n-homogeneous*. A point set which is 1-homogeneous is said to be *homogeneous*.

A point set M is said to be *nearly n-homogeneous* if for any n points $x_1, x_2, \cdots, x_n$ of M and any n open subsets[2] $D_1, D_2, \cdots, D_n$ of M there exist n points $y_1, y_2, \cdots, y_n$ of $D_1, D_2, \cdots, D_n$ respectively and a homeomorphism T of M onto itself such that $T(x_1+x_2+\cdots+x_n)=y_1+y_2+\cdots+y_n$. A set which is nearly 1-homogeneous is said to be *nearly homogeneous*.

A point set M is said to be *nearly homogeneous over a subset H of M* if for any point x of H and any open subset D of M there is a homeomorphism of M onto itself that carries x into a point of D.

It is obvious that every n-homogeneous set is nearly n-homogene-

Presented to the Society, December 29, 1953; received by the editors May 29, 1953.

[1] Some properties of such continua are given in [8] and [18].

[2] A subset of M open relative to M is called an *open subset of M*.

ous and that every nearly n-homogeneous set is nearly homogeneous.

2. **Homogeneous continua in a metric space.** Theorems 1–6 are proved on the basis of R. L. Moore's[3] Axioms 0 and 1_3. Hence these theorems hold true in any metric space.

THEOREM 1. *Every n-homogeneous point set M is homogeneous.*

PROOF. Suppose M is not homogeneous. Then there are two points x_1 and y_1 of M such that there does not exist a homeomorphism of M onto itself that carries x_1 into y_1. Let $x_2, x_3, \cdots, x_n$ be distinct points of M different from x_1 and y_1. There is a homeomorphism T of M onto itself such that $T(x_1+x_2+\cdots+x_n)=y_1+x_2+x_3+\cdots+x_n$. Since $T(x_1)\neq y_1$, then there is a positive integer i_1 $(1<i_1\leqq n)$ such that $T(x_1)=x_{i_1}$. If $T(x_{i_1})\neq y_1$, then there is a positive integer i_2 $(1<i_2\leqq n)$ different from i_1 such that $T(x_{i_1})=x_{i_2}$. This process can be continued to obtain a positive integer r such that $1<i_r\leqq n$ and $T(x_{i_r})=y_1$. Since T^{r+1} is a homeomorphism of M onto itself carrying x_1 into y_1, then M is homogeneous.

THEOREM 2. *If $n>1$ and the compact continuum M is indecomposable under index n, then M is not nearly homogeneous.*

PROOF. Suppose, on the contary, that M is nearly homogeneous. By a theorem proved by Swingle [**18**, Theorem 2], M is the finished sum of n indecomposable continua $M_1, M_2, \cdots, M_n$ such that M_1 and M_2 have a point x in common. Let D be an open subset of M which intersects M_1 and does not intersect $M_2+M_3+\cdots+M_n$. There is a homeomorphism T of M onto itself that carries x into a point of D. Then M is the finished sum of the indecomposable continua $T(M_1), T(M_2), \cdots, T(M_n)$. Since $T(x)$ lies in both $T(M_1)$ and $T(M_2)$ and does not lie in $M_2+M_3+\cdots+M_n$, then one of the continua $T(M_1)$ and $T(M_2)$ is not one of the continua $M_1, M_2, \cdots, M_n$. This is contrary to Theorem 2 of [**8**].

THEOREM 3. *If $n>1$ and the compact continuum M is nearly homogeneous over a set H consisting of n points about which M is irreducible, then there is a positive integer r not greater than n such that M is indecomposable under index r.*

PROOF. An inductive argument will be used. It will first be shown that Theorem 3 holds true if $n=2$. Let x and y be the two points of H. Suppose that M is the finished sum of three continua M_1, M_2, and M_3. Consider the case in which $x+y$ is a subset of M_1+M_2. Then

[3] Axiom 1_3 is the first three parts of Axiom 1 as stated in [**17**].

M_1+M_2 is not connected and M_3 intersects each of the continua M_1 and M_2. There is a homeomorphism T of M onto itself that carries x into a point of $M-(M_1+M_2)$. Then one of the continua M_1+M_3 and M_2+M_3 contains both of the points $T(x)$ and $T(y)$. This is impossible since each of these continua is a proper subset of M and M is irreducible between the points $T(x)$ and $T(y)$. Thus the supposition that M is the finished sum of three continua has led to a contradiction. Hence Theorem 3 holds true if $n=2$.

Now let k be a positive integer such that Theorem 3 holds true if $n=k$. For the remainder of this argument it will be assumed that H consists of $k+1$ points. Suppose that there is a collection G consisting of $k+2$ continua whose finished sum is M. Then there is a continuum X of G such that the set[4] $M-(G-X)^*$ contains no point of H. Let z be a point of H. There is a homeomorphism T of M onto itself that carries z into a point of $M-(G-X)^*$. Since $T(H)-T(z)$ consists of k points, then some proper subcontinuum M' of M contains $T(H)-T(z)$ but not $T(z)$. Since $M'+(G-X)^*$ is a proper subset of M containing H, then $M'+(G-X)^*$ is not connected. Hence some continuum L of the collection $G-X$ does not intersect M'. Since $(G-L)^*$ is a proper subset of M containing $T(H)$, then $(G-L)^*$ is not connected. Hence $(G-L)^*$ is the sum of two mutually separated sets L_1 and L_2 such that L_1 contains the continuum M'. Since L_1+L is a proper subcontinuum of M, then L_2 contains $T(z)$. Consequently L_2 contains the continuum X. Since the closed set $(G-X)^*$ contains H, then $(G-X)^*$ is the sum of two mutually separated sets X_1 and X_2 such that X_1 contains the continuum L_1+L. Hence X_1+X contains $T(H)$. This is impossible since X_1+X is a proper subcontinuum of M and M is irreducible about $T(H)$. Thus the supposition that M is the finished sum of $k+2$ continua has led to a contradiction. This completes the inductive argument.

THEOREM 4. *If the nondegenerate compact continuum M is nearly homogeneous and is irreducible about some finite set, then M is indecomposable.*

This theorem follows from Theorems 2 and 3.

COROLLARY. *If the nondegenerate compact continuum M is nearly homogeneous and is homeomorphic with each of its nondegenerate subcontinua, then M is indecomposable.*[5]

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[4] The collection of all elements of G different from X is denoted by $G-X$, and the sum of the sets of $G-X$ is denoted by $(G-X)^*$.

[5] If M is a homogeneous plane continuum this follows from [11, Theorem 2] and [19, Theorem 3].

THEOREM 5. *If $n>1$, then in order that the compact continuum M should be indecomposable under index n it is necessary and sufficient that* (1) *M should contain a set H consisting of n points such that M is asposyndetic*[6] *at each point x of H with respect to each point of $H-x$ and* (2) *M should not contain $n+1$ such points.*

PROOF OF NECESSITY. By [18, Theorem 2], M is the finished sum of n indecomposable continua $M_1, M_2, \cdots, M_n$. Let H be a set consisting of n points $x_1, x_2, \cdots, x_n$ such that for each i $(i \leqq n)$, x_i is a point of M_i and does not belong to M_j $(j \neq i)$. Then H is aposyndetic at each point of H with respect to each other point of H. Now suppose that there exists a set K consisting of $n+1$ points of M such that M is aposyndetic at each point of K with respect to each other point of K. Then there is a positive integer i such that M_i contains two points y_1 and y_2 of K. There is a continuum L and an open subset U of M such that $M-y_1 \supset L \supset U \supset y_2$. This is contrary to Lemma 1.2 of [7].

PROOF OF SUFFICIENCY. Since M does not contain $n+1$ points such that M is aposyndetic at each of them with respect to each of the others, then M is not the finished sum of $n+1$ continua. Hence there is a positive integer r not greater than n such that M is indecomposable under index r. It follows from the necessary condition proved above that $r=n$.

THEOREM 6. *If $n>1$ and the compact continuum M is nearly homogeneous and does not contain n points such that M is aposyndetic at each of them with respect to each of the others, then M is indecomposable.*

This theorem follows from Theorems 2 and 5.

3. **Homogeneous continua in the plane.** The following definition is used in the proof of Theorem 7. A continuum M is said to have property W if every open subset of M contains both a point of a proper subset K of M and a point of $M-K$ such that K is an irreducible cutting[7] of the plane.

THEOREM 7. *If the bounded plane continuum M is nearly homogeneous and separates the plane into a finite number of connected domains, then M is the boundary of each of its complementary domains.*

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[6] A continuum M is said to be *aposyndetic* at a point x of M with respect to a point y of M if there exists a subcontinuum K of M and an open subset U of M such that $M-y \supset K \supset U \supset x$.

[7] A point set K is said to be an *irreducible cutting* of the plane if K separates the plane and no proper subset of K does so. Every such set is closed [20, page 43].

LEMMA 7.1. *If the bounded plane continuum M has property W and contains no domain, then every domain intersecting M intersects infinitely many complementary domains of M.*

PROOF OF LEMMA 7.1. Suppose that this lemma is not true. Then there exists a positive integer n such that (1) every domain intersecting M intersects at least n complementary domains of M and (2) some domain D intersects M and does not intersect $n+1$ complementary domains of M. There is a proper subset K of M such that D intersects both K and $M-K$ and K is an irreducible cutting of the plane. There is a connected domain D' lying in D and intersecting M but not K. There is a complementary domain E of K intersecting D but not D'. Hence some complementary domain E' of M lies in E and intersects D but not D'. Since D' intersects n complementary domains of M, then D intersects $n+1$ complementary domains of M. This involves a contradiction.

PROOF OF THEOREM 7. Suppose that some complementary domain of M does not have M as its boundary. Since the boundary of this domain separates the plane, it follows from two theorems proved by Kuratowski [**13**, Theorems 2 and 4] that some proper subcontinuum K of M is an irreducible cutting of the plane. Let D be any open subset of M and let x be a limit point of $M-K$ lying in K. There is a homeomorphism T of M onto itself that carries x into a point of D. It follows from a theorem proved by Brouwer [**5**, p. 425] that $T(K)$ is an irreducible cutting of the plane. Since $T(K)$ is a proper subset of M and $T(x)$ is a limit point of $M-T(K)$, it follows that M has property W. Since a plane domain is invariant under a homeomorphism [**4**], it follows that M contains no domain. Hence by Lemma 7.1, every domain intersecting M intersects infinitely many complementary domains of M. This is a contradiction since M has only a finite number of complementary domains.

THEOREM 8. *If the bounded plane continuum M is nearly homogeneous and has more than two but only a finite number of complementary domains, then M is indecomposable.*[8]

PROOF. Suppose that M is decomposable. By Theorem 7, M is the boundary of each of its complementary domains. It follows from an argument given by Kuratowski that M is indecomposable under index 2. (See [**14**, Theorem 3] and the argument thereof.) This is contrary to Theorem 2.

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[8] The author does not know whether there exists a continuum which satisfies the hypothesis of this theorem.

THEOREM 9. *If $n>1$ and the bounded plane continuum M is nearly n-homogeneous and is not a continuous curve, then M is indecomposable*[9].

INDICATION OF PROOF. Suppose that M is the finished sum of five continua $M_1, M_2, \cdots, M_5$. Let $K_1, K_2, \cdots, K_5$ be circular domains such that (1) $\overline{K}_1, \overline{K}_2, \cdots, \overline{K}_5$ are mutually exclusive, (2) for each i ($i \leqq 5$), K_i intersects M_i, and (3) if $j \neq i$, $\overline{K}_i$ does not intersect M_j. By [17, Theorem 47, p. 132], there exist two open subsets U_1 and U_2 of M, a continuum H, and a sequence of mutually exclusive continua $H_1, H_2, H_3, \cdots$ such that (1) U_1 and U_2 are mutually exclusive, (2) for each n, H_n intersects both U_1 and U_2 and is a component of $M-(U_1+U_2)$, and (3) the sequence $H_1, H_2, H_3, \cdots$ converges to H. Since M is nearly n-homogeneous, there exists a homeomorphism T of M onto itself that carries some point x_1 of H into a point of K_1 and some point x_2 of H into a point of K_2. There exist open subsets R_1 and R_2 of M containing x_1 and x_2 respectively such that $T(R_1) \subset K_1$ and $T(R_2) \subset K_2$. Since infinitely many of the sets $H_1, H_2, H_3, \cdots$ intersect both R_1 and R_2, then infinitely many of the sets $T(H_1)$, $T(H_2)$, $T(H_3), \cdots$ intersect both K_1 and K_2. Consequently there exists an infinite sequence $H_1', H_2', H_3', \cdots$ of mutually exclusive subcontinua of M such that (1) for each n, H_n' intersects both $\overline{K}_1$ and $\overline{K}_2$ and is a component of $M-M\cdot(K_1+K_2)$, and (2) for each i ($i \leqq 2$), some arc p_iq_i on the boundary J_i of K_i contains the sets $J_i\cdot H_1', J_i\cdot H_2', J_i\cdot H_3', \cdots$ in the order named from p_i to q_i. For each n, there is an arc a_nb_n irreducible from $\overline{K}_1$ to $\overline{K}_2$ such that (1) a_n is between $J_1\cdot H_n'$ and $J_1\cdot H_{n+1}'$ on p_1q_1 and b_n is between $J_2\cdot H_n'$ and $J_2\cdot H_{n+1}'$ on p_2q_2 and (2) a_nb_n does not intersect M. For each n, let D_n denote the complementary domain of $M+\overline{K}_1+\overline{K}_2$ which contains $a_nb_n-(a_n+b_n)$. Since the domains $D_1, D_2, D_3, \cdots$ are mutually exclusive, it can be shown by an argument quite similar to the one used in Case 1 of the proof of Theorem 9 of [6] that no one of the sets $\overline{K}_3$, $\overline{K}_4$, and $\overline{K}_5$ intersects more than two of these domains. Hence one of the domains $D_1, D_2, D_3, \cdots$ does not intersect $\overline{K}_3+\overline{K}_4+\overline{K}_5$. From this it follows that if $i \neq j$ ($i, j \leqq 5$), there is a complementary domain Z of $M+\overline{K}_i+\overline{K}_j$ and an arc t not intersecting M such that (1) Z does not intersect any of the sets $\overline{K}_1, \overline{K}_2, \cdots, \overline{K}_5$ and (2) t is irreducible from $\overline{K}_i$ to $\overline{K}_j$ and lies, except for its end points, in Z. Let $Z_1, Z_2, \cdots, Z_{10}$ be ten such domains corresponding to the ten ways the integers i and j can be chosen. An argument similar to the one used in the proof of Theorem 5 of [7] can be used to show that some two of these ten domains have a point in common. Consider the case in which (1) Z_1 and Z_2 have a point in common, (2) Z_1 is a comple-

[9] Bing [2, Theorem 14] has shown that such a continuum exists.

mentary domain of $M+\overline{K}_1+\overline{K}_2$ and for each i $(i\leqq 2)$, $\overline{K}_i-M\cdot\overline{K}^i$ contains a boundary point of Z_1, and (3) Z_2 is a complementary domain of $M+\overline{K}_3+\overline{K}_r$, where $r\neq 3$, and for each i $(i=3, r)$, $\overline{K}_i-M\cdot\overline{K}_i$ contains a boundary point of Z_2. Then Z_1 contains Z_2 together with every boundary point of Z_2 that belongs to $\overline{K}_3-M\cdot\overline{K}_3$. Hence Z_1 intersects $\overline{K}_3$. As this involves a contradiction, it follows that there is a positive integer k less than five such that M is indecomposable under index k. Since M is nearly homogeneous, then by Theorem 2, M is indecomposable.

THEOREM 10. *If $n>1$ and the bounded plane continuum M is n-homogeneous, then M is a simple closed curve.*

PROOF. Suppose that M is indecomposable. Let $x_1, x_2, \cdots, x_n$ be n points lying in the same composant of M and let $y_1, y_2, \cdots, y_n$ be n points of M such that no composant of M contains all of them. Since some proper subcontinuum of M contains $x_1+x_2+\cdots+x_n$ and no proper subcontinuum of M contains $y_1+y_2+\cdots+y_n$, there does not exist a homeomorphism of M onto itself that carries $x_1+x_2+\cdots+x_n$ into $y_1+y_2+\cdots+y_n$. As this involves a contradiction, then M is decomposable. Since M is nearly n-homogeneous, it follows from Theorem 9 that M is a continuous curve. By Theorem 1, M is homogeneous. Mazurkiewicz [**15**] has shown that every homogeneous bounded plane continuous curve is a simple closed curve. Hence M is a simple closed curve.

4. **Questions and remarks.** So far as the author knows, the following two questions have not as yet been answered. (1) Does there exist a homogeneous bounded plane continuum which separates the plane and is not a simple closed curve? (See [**12**, p. 149] and [**3**, p. 49].) Cohen [**9**] has shown that if such a continuum exists, then it does not contain a simple closed curve. (2) Is every homogeneous decomposable bounded plane continuum M a simple closed curve? Jones [**10**] has shown that the answer is in the affirmative if M is aposyndetic.

There does exist a *nearly* homogeneous bounded plane dendron. Such a set can be easily constructed. The set of all end points of any such dendron M would be everywhere dense in M.

There exists a decomposable continuum which satisfies the hypothesis of Theorem 3. For each i $(i\leqq 2)$, let H_i be a set indicated by Fig. 1 of [**6**] such that the common part of H_1 and H_2 is the point O. Let K_i be a composant of H_i not containing O and let x_i be a point of K_i. The decomposable continuum H_1+H_2 is irreducible between x_1 and x_2 and is nearly homogeneous over x_1+x_2.

Anderson [1] has shown that there exists a one-dimensional compact continuous curve M such that for each n, M is strongly n-homogeneous. If $n>3$, then no simple closed curve is strongly n-homogeneous. Hence it follows from Theorem 10 that if $n>3$, then no bounded *plane* continuum is strongly n-homogeneous.

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