PROOF OF THE WELL-ORDERING OF CARDINAL NUMBERS

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It is well known that the class of cardinal numbers is well-ordered. But the proofs that we know are long ones, using Zorn's Theorem and the cumbersome theory of ordinal numbers. In this paper we give a very short proof of this theorem using both the Axiom of Choice and Zorn's Theorem.

By the theorem of Bernstein-Cantor we know that the cardinal numbers form an order class. If we prove that every family of cardinal numbers has a first element it will follow that it is totally ordered (if not, a set of two incomparable elements would not have a first element) and indeed, well-ordered.

We shall use the notations and terminology of Bourbaki.

THEOREM 1. *Let* $(\aleph_i)_{i\in I}$ *be a family of cardinal numbers of subsets* A_i *of a set E. Then there exists an* $i_0\in I$ *such that* $\aleph_{i_0}\leqq\aleph_i$ *for every* $i\in I$.

PROOF. We shall have our result if we prove that for every $i\in I$ there exists a one-to-one function f_i from A_{i_0} into A_i.

Let us take the cartesian product $A=\prod_{i\in I}A_i$ and let us consider the class $\mathcal{B}$ of subsets B of A such that

(*) $x=(x_i)\in B$, $y=(y_i)\in B$, and $x\neq y$ implies $x_i\neq y_i$ for every $i\in I$.

It is immediate that $\mathcal{B}$ is inductive when ordered by inclusion and hence, by Zorn's Theorem, $\mathcal{B}$ has at least one maximal element $\overline{B}$. We assert that there exists an $i_0\in I$ such that $pr_{i_0}(\overline{B})=A_{i_0}$: for if $pr_i(\overline{B})\neq A_i$ for every $i\in I$ we could take (by the Axiom of Choice) an element a_i in every $A_i-pr_i(\overline{B})$ and set $\overline{B}\cup\{a\}$ where $a=(a_i)$ would strictly contain $\overline{B}$ and it would still satisfy (*) and so $\overline{B}$ would not be maximal. If $pr_{i_0}(\overline{B})=A_{i_0}$ we define the one-to-one function f_i from A_{i_0} into A_i by $x_{i_0}\in A_{i_0}\rightarrow f_i(x_{i_0})=x_i=pr_ix$ where x is the point of $\overline{B}$ such that $pr_{i_0}x=x_{i_0}$; by (*) this point $x\in\overline{B}$ is unique and f_i is one-to-one $(x_{i_0}\neq y_{i_0}\rightarrow x\neq y\rightarrow x_i\neq y_i)$. Q.E.D.

COROLLARY. *The cardinal numbers of subsets of a set E form a well-ordered set.*

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