

## PROOF OF THE WELL-ORDERING OF CARDINAL NUMBERS

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It is well known that the class of cardinal numbers is well-ordered. But the proofs that we know are long ones, using Zorn's Theorem and the cumbersome theory of ordinal numbers. In this paper we give a very short proof of this theorem using both the Axiom of Choice and Zorn's Theorem.

By the theorem of Bernstein-Cantor we know that the cardinal numbers form an order class. If we prove that every family of cardinal numbers has a first element it will follow that it is totally ordered (if not, a set of two incomparable elements would not have a first element) and indeed, well-ordered.

We shall use the notations and terminology of Bourbaki.

**THEOREM 1.** *Let  $(\aleph_i)_{i \in I}$  be a family of cardinal numbers of subsets  $A_i$  of a set  $E$ . Then there exists an  $i_0 \in I$  such that  $\aleph_{i_0} \leq \aleph_i$  for every  $i \in I$ .*

**PROOF.** We shall have our result if we prove that for every  $i \in I$  there exists a one-to-one function  $f_i$  from  $A_{i_0}$  into  $A_i$ .

Let us take the cartesian product  $A = \prod_{i \in I} A_i$  and let us consider the class  $\mathcal{B}$  of subsets  $B$  of  $A$  such that

(\*)  $x = (x_i) \in B, y = (y_i) \in B$ , and  $x \neq y$  implies  $x_i \neq y_i$  for every  $i \in I$

It is immediate that  $\mathcal{B}$  is inductive when ordered by inclusion and hence, by Zorn's Theorem,  $\mathcal{B}$  has at least one maximal element  $\bar{B}$ . We assert that there exists an  $i_0 \in I$  such that  $pr_{i_0}(\bar{B}) = A_{i_0}$ : for if  $pr_i(\bar{B}) \neq A_i$  for every  $i \in I$  we could take (by the Axiom of Choice) an element  $a_i$  in every  $A_i - pr_i(\bar{B})$  and set  $\bar{B} \cup \{a\}$  where  $a = (a_i)$  would strictly contain  $\bar{B}$  and it would still satisfy (\*) and so  $\bar{B}$  would not be maximal. If  $pr_{i_0}(\bar{B}) = A_{i_0}$  we define the one-to-one function  $f_i$  from  $A_{i_0}$  into  $A_i$  by  $x_{i_0} \in A_{i_0} \rightarrow f_i(x_{i_0}) = x_i = pr_i x$  where  $x$  is the point of  $\bar{B}$  such that  $pr_{i_0} x = x_{i_0}$ ; by (\*) this point  $x \in \bar{B}$  is unique and  $f_i$  is one-to-one ( $x_{i_0} \neq y_{i_0} \rightarrow x \neq y \rightarrow x_i \neq y_i$ ). Q.E.D.

**COROLLARY.** *The cardinal numbers of subsets of a set  $E$  form a well-ordered set.*

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