TWO THEOREMS ON ALMOST SURE CONVERGENCE

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1. Introduction. In [1] the author used two theorems which are presented for their fairly general interest in probability theory. Both are closely related to the theory of martingales, in fact the second is a straightforward generalization of the semi-martingale convergence theorem due to Doob [2]. For a detailed discussion of martingales see [2].

2. A convergence theorem. In this section we use a lemma due to Loève [3]. We state it here as

**Lemma 1.** Let \( \{V_n\} \) be a sequence of random variables such that
\[
\sum_{n=1}^{\infty} E\{V_n^2\} < \infty.
\]
Then \( \sum_{j=1}^{\infty} \left[ V_j - E\{V_j | V_1, \ldots, V_{j-1}\} \right] \) converges a.s. (almost surely).

Let \( \{X_n\} \) be a sequence of random variables and define the sequence \( \{Y_n\} \) by \( Y_1 = X_1, \ Y_n = X_n - X_{n-1} \) for \( n > 1 \). Denote the variance of \( Y_n \) by \( \sigma_n^2 \). Then we have

**Lemma 2.** If \( \sum_{n=1}^{\infty} \sigma_n^2 < \infty \), then \( X_n - \sum_{j=1}^{n} E\{X_j - X_{j-1} | X_1, \ldots, X_{j-1}\} \) converges a.s.

**Proof.** Let \( V_j = Y_j - E\{Y_j | Y_1, \ldots, Y_{j-1}\} \). Then \( E\{V_j^2\} \leq \sigma_j^2 \) and Lemma 1 applies. But \( E\{V_j | V_1, \ldots, V_{j-1}\} \) = 0 a.s., since the \( \sigma \)-field determined by \( V_1, \ldots, V_{j-1} \) coincides with the \( \sigma \)-field determined by \( X_1, \ldots, X_{j-1} \). Hence \( \sum_{j=1}^{\infty} V_j \) converges a.s. and this is clearly equivalent with the conclusion of the lemma.

**Theorem 1.** Let \( \{X_n\} \) be a sequence of random variables such that
\[
\sum_{n=1}^{\infty} \sigma_n^2 < \infty.
\]

(2.1) For every \( \epsilon > 0 \) there exists a nonnegative integer valued random variable \( N(\epsilon) \) such that \( n \geq N(\epsilon) \) and \( |X_n| \leq \epsilon \) implies
\[
X_n E\{Y_{n+1} | Y_1, \ldots, Y_n\} \leq 0 \text{ a.s.}
\]
Then a necessary and sufficient condition that \( X_n \) converge a.s. is

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3 Numbers in brackets refer to the bibliography at the end of the paper.

253
that $E\{ Y_n \mid Y_1, \ldots, Y_{n-1} \}$ converge a.s. to 0.

Proof. We have $Y_n = (X_n - \sum_{j=1}^{n-1} E\{ Y_j \mid Y_1, \ldots, Y_{j-1} \}) - (X_{n-1} - \sum_{j=1}^{n-1} E\{ Y_j \mid Y_1, \ldots, Y_{j-1} \}) + E\{ Y_n \mid Y_1, \ldots, Y_{n-1} \}$. Hence by applying (2.1) and Lemma 2 we have $E\{ Y_n \mid Y_1, \ldots, Y_{n-1} \}$ converges to 0 a.s. if and only if $Y_n$ does. This shows necessity. Now suppose (2.1) and (2.2) hold and $E\{ Y_n \mid Y_1, \ldots, Y_{n-1} \}$ converges to 0 a.s. Then so does $Y_n$. Next we show

$$P\{\lim_{n \to \infty} X_n = +\infty\} = P\{\lim_{n \to \infty} X_n = -\infty\} = 0.$$ 

For suppose $\{X_n\}$ is a sample sequence for which the conclusion of Lemma 2 and (2.2) holds. Now if $\lim_{n \to \infty} X_n = \infty$, then for sufficiently large $n$, $E\{ Y_{n+1} \mid Y_1, \ldots, Y_n \} \leq 0$. But then we would have $\lim_{n \to \infty} [X_n - \sum_{j=1}^{n-1} E\{ Y_j \mid Y_1, \ldots, Y_{j-1} \}] = \infty$ which is impossible by Lemma 2. Similarly if $\lim_{n \to \infty} X_n = -\infty$. Now suppose $P\{\lim_{n \to \infty} X_n \text{ exists finite}\} < 1$. Then there exists a set of positive probability such that for every sample sequence $\{X_n\}$ in the set we have:

(i) (2.2) holds.
(ii) $X_n - \sum_{j=1}^{n-1} E\{ Y_j \mid Y_1, \ldots, Y_{j-1} \}$ converges to a finite number.
(iii) $\lim \inf_{n \to \infty} X_n < \lim \sup_{n \to \infty} X_n$.
(iv) $Y_n$ converges to 0.

Let $\{X_n\}$ be such a sequence and assume $\lim \sup_{n \to \infty} X_n > 0$. (A similar argument applies when $\lim \sup_{n \to \infty} X_n < 0$.) Choose numbers $a$ and $b$ with

$$0 < a, \quad \lim \inf_{n \to \infty} X_n < a < b < \lim \sup_{n \to \infty} X_n.$$ 

Now choose $N$ so large that the following conditions are satisfied:

(i) $N > N(a/2)$ where $N(a/2)$ satisfies (2.2);
(ii) $|Y_n| < a/2$ for $N < n$;
(iii) $|X_m - X_n - \sum_{j=n+1}^{m} E\{ Y_j \mid Y_1, \ldots, Y_{j-1} \}| < b - a$ for $N \leq n < m$.

and choose $m$ and $n$ with $N \leq n < m$ such that:

(i) $X_n < a, X_m > b$;
(ii) $a \leq X_j \leq b$ for $n < j < m$.

Now $X_m - X_n > b - a$ and $X_n > a/2$. But then $\sum_{j=n+1}^{m} E\{ Y_j \mid Y_1, \ldots, Y_{j-1} \} \leq 0$ which is clearly a contradiction and thus proves the theorem.

3. Extension of a martingale theorem. In [2] Doob has proved the following theorem. If $\{X_n\}$ is a sequence of random variables with:

(i) $\sup_{n} E\{|X_n|\} < \infty$;
(ii) $E\{X_{n+1} - X_n \mid X_1, \ldots, X_n\} \leq 0 \text{ a.s.},$
then $X_n$ converges a.s. The following generalization is an immediate consequence of the theorem quoted.

**Theorem 2.** Let $\{X_n\}$ be a sequence of random variables satisfying

\[
\text{(3.1) } \sup_n E\{|X_n|\} < \infty,
\]

\[
\text{(3.2) } \sum_{n=1}^{\infty} E\{E\{X_{n+1} - X_n \mid X_1, \ldots, X_n\}\} < \infty.
\]

Then $X_n$ converges a.s.

As usual we define $X^+$ by $X^+ = \left[ X+ \mid X \right] / 2$. One interesting aspect of Theorem 2 is that (3.2) is in a sense the best condition of its kind. To see this let $\{q_n\}$, $n = 1, 2, \ldots$, be a sequence of numbers with $0 \leq q_n \leq 1$. We construct a nonstationary Markov chain $\{X_n\}$ by letting $X_0$ assume the values $\pm 1$ with probability $1/2$ respectively and letting $X_n$ assume the values $\pm 1$ with the transition probabilities

\[
P\{X_n = 1 \mid X_{n-1} = 1\} = p_n, \quad P\{X_n = -1 \mid X_{n-1} = 1\} = q_n,
\]

\[
P\{X_n = 1 \mid X_{n-1} = -1\} = q_n, \quad P\{X_n = -1 \mid X_{n-1} = -1\} = p_n,
\]

where $p_n = 1 - q_n$. Now define the sequence of events $\{A_n\}$ by $A_n = \{(X_{n-1} = 1, X_n = -1) \text{ or } (X_{n-1} = -1, X_n = 1)\}$. Now it is clear that $X_n$ converges a.s. if and only if $P\{A_n \text{ occurs infinitely often} \} = 0$. By direct computation it is easily verified that the $A_n$ are independent events and that $P\{A_n\} = q_n$. Hence it follows from the Borel-Cantelli lemma that $X_n$ converges a.s. if and only if $\sum_{n=1}^{\infty} q_n < \infty$. Furthermore, it is easily seen that $E\{E\{X_{n+1} - X_n \mid X_1, \ldots, X_n\}\} = q_{n+1}$. Hence we find that for the family of Markov chains thus defined (3.2) is both necessary and sufficient, so that no weakening of (3.2) involving the summands is possible.

**References**


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