SETS WHICH SEPARATE SPHERES

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An extended version of the Jordan curve theorem [1] states that if X and Y are respectively an n-manifold and an $(n+1)$-sphere and X is contained in Y, then $Y-X$ is the union of two disjoint, open, connected sets each having X as point set boundary. It is shown here that it is possible to relax the requirement that X be an n-manifold in such a way that the conclusions continue to hold. Actually, topological conditions on X will be given that are necessary and sufficient for X to separate the $(n+1)$-sphere S^{n+1} in the stated manner.

A cohomology theory is assumed to be defined on the category of compact pairs and to satisfy the continuity property as well as the axioms of Eilenberg and Steenrod [2]. Spanier [3] has given one of several ways of showing the existence of such a theory having an arbitrary abelian coefficient group G (discrete).

For such a cohomology theory an n-manifold X contained in S^{n+1} has the following well known properties which are assumed here without proof:

(1) $H^n(X)$ is isomorphic to G;

(2) $H^n(A)=0$ for every closed proper subset A of X.

It will be shown that for any closed subset X of S^{n+1}, (1) and (2) are necessary and sufficient conditions for $S^{n+1}-X$ to be the union of two disjoint, open, connected sets each having X as point set boundary.

For $n=0$, $H^n(X)$ is taken to be the reduced zero-dimensional group.

The material in §1 is contained in the Tulane University lecture notes of A. D. Wallace.

1. If (X, A) is a compact pair the inclusion map

$$i: A \rightarrow X$$

induces the homomorphism

$$i^*: H^n(X) \rightarrow H^n(A).$$

An element e in $H^n(A)$ is said to be extendable to the element e' in $H^n(X)$ if $e=i^*(e')$. If attention is not directed to the specific element e', the situation can be described by saying that e can be extended to X. For an element e' in $H^n(X)$ the element $i^*(e')$ is de-

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noted by $e'|A$. The set A is said to support e' if $e'|A$ is not zero. If e' is not zero but $e'|A$ is zero for every closed proper subset A of X, then X is said to be a minimal support for e'.

For a cohomology theory satisfying the continuity property on the category of compact pairs, every compact triad $(X; X_1, X_2)$ is a proper triad and the Mayer-Vietoris sequence [2, p. 43],

$$\xrightarrow{\phi} H^n(X_1) + H^n(X_2) \xrightarrow{\psi} H^n(X_1 \cap X_2) \xrightarrow{\Delta} H^{n+1}(X) \xrightarrow{\phi}$$

is exact. It will be recalled that $X = X_1 \cup X_2$ and that for an element (e_1, e_2) in $H^n(X_1) + H^n(X_2)$,

$$\psi(e_1, e_2) = e_1 | X_1 \cap X_2 - e_2 | X_1 \cap X_2.$$

THEOREM 1.1 (THE EXTENSION THEOREM). *If (X, A) is a compact pair and e an element of $H^n(A)$, then e can be extended to $H^n(P)$ where P is the closure of an open set containing A.*

This is an easy consequence of the continuity property (compare [4, p. 247]).

THEOREM 1.2. *Let $X = X_1 \cap X_2$, $Y = X_1 \cup X_2$ where Y is compact and X_1 and X_2 are closed in Y. If an element e in $H^n(X)$ can be extended to e_1 in $H^n(X_1)$ and to e_2 in $H^n(X_2)$, then it can be extended to an element e' in $H^n(Y)$ so that $e'|X_1 = e_1$ and $e'|X_2 = e_2$.*

PROOF. Consider the Mayer-Vietoris sequence

$$\xrightarrow{\Delta} H^n(Y) \xrightarrow{\phi} H^n(X_1) + H^n(X_2) \xrightarrow{\psi} H^n(X) \xrightarrow{\Delta}$$

and let e, e_1 and e_2 be as in the statement of the theorem. Then $\psi(e_1, e_2) = e_1|X - e_2|X = e - e = 0$ so that by exactness $(e_1, e_2) = \phi(e')$ for some e' in $H^n(Y)$. Now

$$e' | X_1 = e_1,$$
$$e' | X_2 = e_2,$$
$$e' | X = (e' | X_1) | X = e_1 | X = e$$

and the conclusions of the theorem are verified.

DEFINITION. *Let X and M be closed sets in the compact space Y and let e be an element of $H^n(X)$ which cannot be extended to $X \cup M$ but can be extended to $X \cup N$ for every closed proper subset N of M. Then M is called an irreducible membrane for e relative to Y.*

THEOREM 1.3. *If X is a closed subset of the compact space Y and e an element of $H^n(X)$ which cannot be extended to Y, then there is an* *irreducible membrane for e relative to Y.*

Using the extension theorem the proof is as in [**5**, p. 32].

THEOREM 1.4. *If (Y, X) is a compact pair and M is an irreducible membrane for an element e of $H^n(X)$ relative to Y then:*

(a) $\mathrm{Cl}\,(M-X)=M$;

(b) *$X\cap M$ supports e.*

PROOF. (a) if $X\cap M$ contains a nonempty set V open in M let $N=M-V$. Then e can be extended to $X\cup M=X\cup N$.

(b) If $e|X\cap M=0$, then this element can be extended to e in $H^n(X)$ and to the zero element in $H^n(M)$ and then by 1.2 can be extended to $X\cup M$ in such a way as to provide an extension of e to $X\cup M$.

THEOREM 1.5. *If X is a closed subset of S^{n+1} and M is an irreducible membrane for an element e of $H^n(X)$, then $M-X$ is open.*

PROOF. Suppose $M-X$ is not open. Then there is a point x of $M-X$ such that every neighborhood of x intersects the complement of $M-X$. This point has a spherical neighborhood U whose boundary $F(U)$ intersects the complement of $M-X$ and whose closure does not intersect X. Let $A=M\cap F(U)$, $N_1=M-U$, $N_2=M\cap\overline{U}$. Then N_1 and N_2 are closed proper subsets of M, $N_1\cup N_2=M$, $N_1\cap N_2=A$. A then is a closed proper subset of the n-sphere $F(U)$. By hypothesis the element e can be extended to an element e' in $H^n(X\cup N_1)$. Also $e'|A=0$ since $H^n(A)=0$. Therefore $e'|A$ can be extended to e' in $H^n(X\cup N_1)$ and to the zero element in $H^n(N_2)$ and then to $X\cup N_1 \cup N_2=X\cup M$ as in Theorem 1.2. But this extension would also be an extension of e to $X\cup M$ which is impossible. Therefore $M-X$ is open.

2. THEOREM 2.1. *Let M be an irreducible membrane for an element e in $H^n(X)$. Then $M-X$ is connected.*

PROOF. Suppose that $M-X=P\cup Q$ where P and Q are non-empty separated sets. Then $\overline{P}\cap Q=\square$ implies that $\overline{P}-P$ is contained in $M\cap X$. Let $N_1=(M\cap X)\cup P$, $N_2=(M\cap X)\cup Q$. Then $\overline{N}_1 =(\mathrm{Cl}\,(M\cap X))\cup\overline{P}=(M\cap X)\cup P\cup(\overline{P}-P)=(M\cap X)\cup P=N_1$. Likewise $\overline{N}_2=N_2$ so that N_1 and N_2 are closed proper subsets of M. Also $N_1\cup N_2=M$, $N_1\cap N_2=M\cap X$. Now e can be extended to $X\cup N_1$ and to $X\cup N_2$ and hence by 1.2 to $X\cup M$ which is impossible. Therefore $M-X$ is connected.

THEOREM 2.2. *If X is a minimal support for an element e of $H^n(X)$*

and M is an irreducible membrane for this element such that $M-X$ is open, then X is the boundary of $M-X$.

PROOF. Let $A=M-X$ and $F(A)$ be the boundary of A. By 1.4, $\overline{A}=M$ and $M\cap X$ supports e. Since X is a minimal support for e it follows that $M\cap X=X$. Since A is open, $F(A)=\overline{A}-A=M-(M-X)$ $=M\cap X=X$.

THEOREM 2.3. *If X is a closed subset of S^{n+1} and is a minimal support for an element e in $H^n(X)$, then there are irreducible membranes M_1 and M_2 for e relative to S^{n+1} such that M_1-X and M_2-X are disjoint and $X=F(M_1-X)=F(M_2-X)$.*

PROOF. By 1.3 there is an irreducible membrane M_1 for e in S^{n+1} and according to 1.5, 2.1, and 2.2, M_1-X is open, connected, and $X=F(M_1-X)$. Now let $N_1=S^{n+1}-(M_1-X)$ and suppose that the element e can be extended to N_1. Let U be a spherical neighborhood whose closure is contained in M_1-X and let $N_2=M_1-U$. Then $N_1\cup N_2=S^{n+1}-U=E^{n+1}$ where E^{n+1} is an $(n+1)$-cell. The element e can now be extended to $X\cup N_2$ since N_2 is a closed proper subset of M_1 and to $X\cup N_1$ by assumption. Since $X=N_1\cap N_2$ it follows that e can be extended to $N_1\cup N_2=E^{n+1}$. But this is impossible since $H^n(E^{n+1})=0$. Therefore e cannot be extended to N_1 and there exists an irreducible membrane M_2 for e relative to N_1 and hence relative to S^{n+1}. As in the case of M_1, M_2-X is open, connected, and, $F(M_2-X)=X$. It is clear from the construction that M_1-X and M_2-X are disjoint.

REMARK. It will be noted that in the proof of 2.3 the assumption that X was a minimal support for e was used only to show that $X=F(M_1-X)=F(M_2-X)$. The existence of the sets M_1 and M_2 depended only on the presence of a nonzero element of $H^n(X)$. From this the necessary part of the next theorem follows easily.

THEOREM 2.4. *If X is a closed subset of S^{n+1}, then for $S^{n+1}-X$ to be connected it is necessary and sufficient that $H^n(X)=0$ (for any nontrivial coefficient group).*

PROOF OF SUFFICIENCY. Suppose that $S^{n+1}-X$ is not connected. Then $S^{n+1}-X=P\cup Q$ where P and Q are open, nonvacuous, and disjoint. Let $N_1=S^{n+1}-P$, $N_2=S^{n+1}-Q$. Then $N_1\cup N_2=S^{n+1}$, $N_1\cap N_2=X$. Consider the following portion of the Mayer-Vietoris sequence:

$$H^n(X)\xrightarrow{\Delta}H^{n+1}(S^{n+1})\xrightarrow{\phi}H^{n+1}(N_1)+H^{n+1}(N_2).$$

The group $H^{n+1}(N_1)+H^{n+1}(N_2)$ is zero since N_1 and N_2 are closed proper subsets of S^{n+1}. Then by exactness the homomorphism Δ maps $H^n(X)$ onto the nontrivial group $H^{n+1}(S^{n+1})$ and $H^n(X)$ cannot be zero.

THEOREM 2.5. *Let (X, A, B) be a compact triple with the boundary of A contained in B. If for a fixed integer p, no nonzero element of $H^p(A)$ can be extended to X, then the homomorphism $i^*:H^p(A)\to H^p(B)$ induced by the inclusion map is an isomorphism into.*

PROOF. Let e be a nonzero element of $H^p(A)$. Since e cannot be extended to X there is by 1.3 an irreducible membrane M for e relative to X. Then by 1.4, $\mathrm{Cl}\ (M-A)=M$ so that $M\cap A=\mathrm{Cl}\ (M-A)\cap A$. Now $M\cap A$ is contained in B since it is contained in $F(A)=\mathrm{Cl}\ (X-A)\cap A$. Also, by 1.4, $e|M\cap A\neq 0$ and it follows that $e|B\neq 0$. Therefore the kernel of i^* is zero.

The following notation is introduced to facilitate the statement of the next proposition. Let X be a closed set in S^{n+1} and denote by Q a fixed component of $S^{n+1}-X$. The remaining components of $S^{n+1}-X$ form a countable set and are denoted by $P_0, P_1, P_2, \cdots$. For each set P_j let $M_j=S^{n+1}-(Q\cup P_j)$ and let $I_j:X\subset M_j$ be the inclusion map. Denote by $\sum H^n(M_j)$ the weak sum of the groups $H^n(M_j)$. Then each element u of $\sum H^n(M_j)$ has a coordinate in each $H^n(M_j)$ and all but a finite number of these coordinates are zero. Define the homomorphism

$$I:\sum H^n(M_j)\to H^n(X)$$

by $I(u)=\sum I_j^*(u_j)$ where u_j is the coordinate of u in $H^n(M_j)$.

THEOREM 2.6. *The homomorphism $I:\sum H^n(M_j)\to H^n(X)$ is an isomorphism onto.*

PROOF. (a) *Kernel of $I=0$.* Let u be an element of $\sum H^n(M_j)$ with coordinates u_j and suppose that $I(u)=0$. Let m be a positive integer large enough that the set $0, 1, 2, \cdots, m$ contains the indices of all nonzero coordinates of u. Then $I(u)=0=\sum_{j=0}^m I_j^*(u_j)$ implies that $I_0^*(u_0)=\sum_{j=1}^m I_j^*(-u_j)=e$. Now the element e can be extended to u_0 in $H^n(M_0)$ and to $\sum_{j=1}^m [-u_j|(X\cup P_0)]$ in $H^n(X\cup P_0)$. Also $M_0\cap(X\cup P_0)=X$ and $M_0\cup(X\cup P_0)=S^{n+1}-Q$ so that, by 1.2, e can be extended to $S^{n+1}-Q$. But by virtue of 2.4, $H^n(S^{n+1}-Q)=0$ since Q is connected. Therefore $e=I_0^*(u_0)=0$. By 2.5, I_0^* is an isomorphism into since X contains the boundary of M_0. Therefore $u_0=0$ and a similar argument shows that $u_j=0$ for $j=1, 2, \cdots, m$. Hence u is the zero of $\sum H^n(M_j)$ and the kernel of $I=0$.

(b) *I is onto.* The proof will be made first for the case where the number of components P_j is finite. If P_0 is the only such set, then $M_0=X$ and I is the identity. Suppose inductively I is onto when there are k components P_j. Then if $Q, P_0, P_1, \cdots, P_k$ are the components of $S^{n+1}-X$, let $X_0=X\cup P_0$ so that $X_0\cup M_0=S^{n+1}-Q$, $X_0 \cap M_0=X$. The Mayer-Vietoris sequence of the triad $(X_0\cup M_0; X_0, M_0)$ contains the part

$$H^n(X_0)+H^n(M_0)\xrightarrow{\psi} H^n(X)\xrightarrow{\Delta} H^{n+1}(S^{n+1}-Q).$$

Now ψ maps $H^n(X_0)+H^n(M_0)$ onto $H^n(X)$ since $H^{n+1}(S^{n+1}-Q)$ is zero.

Next consider the following inclusion maps:

$$\begin{aligned} I_j&:X\subset M_j && (j=0, 1, \cdots, k),\\ I_{0,j}&:X_0\subset M_j && (j=1, 2, \cdots, k),\\ \alpha&:X\subset X_0. \end{aligned}$$

Define the homomorphisms

$$J:\sum_{j=0}^{k} H^n(M_j)\to H^n(X_0)+H^n(M_0),$$

$$I_0:\sum_{j=1}^{k} H^n(M_j)\to H^n(X_0)$$

by $J(u)=(\sum_{j=1}^{k} I_{0,j}^*(u_j), -u_0)$ and $I_0(u')=\sum_{j=1}^{k} I_{0,j}^*(u')$. First note that an element u in $\sum_{j=0}^{k} H^n(M_j)$ can be written uniquely as $u=(u_0, u')$, u_0 in $H^n(M_0)$ and u' an element of $\sum_{j=1}^{k} H^n(M_j)$. Then $J(u)=J(u_0, u')=(I_0(u'), -u)$. Now since $Q, P_1, \cdots, P_k$ are the components of $S^{n+1}-X_0$, I_0 is onto by the induction hypothesis. It is then clear that J is also onto.

Next let u be an element of $\sum_{j=0}^{k} H^n(M_j)$ in the following diagram:

$$\begin{tikzcd} \sum_{j=0}^{k} H^n(M_j) \arrow[rr, "I"] \arrow[dr, "J"'] & & H^n(X) \\ & H^n(X_0)+H^n(M_0) \arrow[ur, "\psi"'] & \end{tikzcd}$$

$I(u)=\sum_{j=0}^{k} I_j^*(u_j)=\sum_{j=1}^{k} I_j^*(u_j)+I_0^*(u_0)=\sum_{j=1}^{k} \alpha^*I_{0,j}^*(u_j)+I_0^*(u_0)$ $=\alpha^*[\sum_{j=1}^{k} I_{0,j}^*(u_j)]-[-I_0^*(u_0)]=[\sum_{j=1}^{k} I_{0,j}^*(u_j)]|X-[-u_0]|X$ $=\psi(\sum_{j=1}^{k} I_{0,j}^*(u_j), -u_0)=\psi J(u)$. Now since ψ and J are both onto and $I=\psi J$, it is seen that I is onto when there are $k+1$ components P_j. This completes the proof for the finite case.

In the general case let e be an element of $H^n(X)$. By 1.1, e can be extended to a set X_1 where X_1 is the union of X and all but a finite number of the sets P_j. Let $Q, P_1, P_2, \cdots, P_k$ be the components of $S^{n+1}-X_1$. In the following diagram i^* is induced by the inclusion map, $I'(u)=\sum_{j=0}^k I_{1,j}^*(u_j)$ where $I_{1,j}: M_j \subset X_1$ is the inclusion, and β is defined by

$$\beta(u)_j = \begin{cases} u_j & \text{for } j \leqq k, \\ 0 & \text{for } j > k. \end{cases}$$

$$\begin{array}{ccc} H^n(X_1) & \xrightarrow{i^*} & H^n(X) \\ {\scriptstyle I'}\uparrow & & \uparrow{\scriptstyle I} \\ \sum_{j=0}^{k} H^n(M_j) & \xrightarrow{\beta} & \sum H^n(M_j) \end{array}$$

By assumption there is an element e' of $H^n(X_1)$ such that $i^*(e')=e$ and since I' is onto, there is an element u of $\sum_{j=0}^k H^n(M_j)$ for which $e'=I'(u)$. Then $e=i^*(e')=i^*I'(u)=i^*[\sum_{j=0}^k I_{1,j}^*(u_j)]=\sum_{j=0}^k i^*I_{1,j}^*(u_j)$ $=\sum_{j=0}^k I_j^*(u_j)=I\beta(u)$. This shows that I is onto in the general case.

Next let $\Lambda=\{\lambda\}$ be the set of indices of the components of $S^{n+1}-X$. Also denote by $\sum_\lambda G_\lambda$ the weak product of a collection of groups indexed by Λ where each group G_λ is isomorphic to the coefficient group G.

THEOREM 2.7. *$H^n(X)$ is isomorphic to $\sum_\lambda G_\lambda$.*

PROOF. As before let $M_j=S^{n+1}-(P_j\cup Q)$ and define $R_j=S^{n+1}-P_j$ and $T_j=S^{n+1}-Q$. Consider the Mayer-Vietoris sequence of the triad $(S^{n+1}; R_j, T_j)$:

$$\xrightarrow{\phi} H^n(R_j)+H^n(T_j) \xrightarrow{\psi} H^n(M_j) \xrightarrow{\Delta} H^{n+1}(S^{n+1}) \xrightarrow{\phi}.$$

By virtue of 2.4, $H^n(R_j)$ and $H^n(T_j)$ are zero since P_j and Q are connected. Also $H^{n+1}(R_j)$ and $H^{n+1}(T_j)$ are zero since R_j and T_j are closed proper subsets of S^{n+1}. By exactness then, $\Delta: H^n(M_j)$ $\approx H^{n+1}(S^{n+1})$. Hence each $H^n(M_j)$ is isomorphic to G and the conclusion follows from 2.6.

THEOREM 2.8. *Let X be a closed subset of S^{n+1}. Then necessary and sufficient conditions that $S^{n+1}-X=P\cup Q$ where P and Q are disjoint, open, connected, and $X=F(P)=F(Q)$ are:*

(1) *$H^n(X)\approx G$ for every coefficient group G,*

(2) *$H^n(A)=0$ for every closed proper subset A of X.*

PROOF. (a) *Necessity*. Condition (1) is a direct consequence of 2.7. As for (2) let A be a closed proper subset of X and set $R=P\cup(X-A)$, $T=Q\cup(X-A)$. Then R and T are respectively between the connected sets P and Q and their closures and hence are connected. Also $R\cup T=[P\cup(X-A)]\cup[Q\cup(X-A)]=P\cup Q\cup(X-A)$ $=(P\cup Q\cup X)-A=S^{n+1}-A$, $R\cap T=[P\cup(X-A)]\cap[Q\cup(X-A)]$ $=X-A\neq\square$. Thus $S^{n+1}-A$, being the union of two intersecting connected sets R and T, is connected and, by 2.4, $H^n(A)=0$. This completes the proof of necessity.

(b) *Sufficiency*. Let the integers be used as coefficient group and then (1) implies that $S^{n+1}-X$ is the union of two components P and Q since otherwise, by 2.7, $H^n(X)$ would not be isomorphic to the coefficient group.

Next choose a nonzero element e of $H^n(X)$. X then is a minimal support for e by virtue of (2). Let M_1 and M_2 be irreducible membranes for e as in 2.3. The set M_1-X, being connected, is contained in the component P, say, of $S^{n+1}-X$. Then since $P\cap M_1=P\cap(M_1-X)=M_1-X$, it is seen that M_1-X is open and closed in P. Therefore $M_1-X=P$ since P is connected. Likewise $M_2-X=Q$ and it follows that $X=F(P)=F(Q)$.

REMARK. Theorem 2.6 is equivalent to a result of Eilenberg and Steenrod [**2**, Theorem 6.10, p. 319]. Also a theorem of Borsuk's [**6**, p. 240] is the same as 2.7 except that Borsuk's proposition involves cohomotopy groups. A comparison shows that for a closed subset of S^{n+1}, the n-dimensional cohomotopy and integral cohomology groups are isomorphic for n greater than 1.

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