

ON THE CONVERGENCE OF ORDERED SETS OF PROJECTIONS¹

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E. R. Lorch has shown [3, p. 223] that a uniformly bounded, naturally ordered sequence of projections in a reflexive Banach space converges to its supremum in the strong topology of operators. In this note it is shown that a uniformly bounded, naturally ordered set of projections in a Banach space X having a weak x -cluster point for each $x \in X$ converges to its supremum in the strong topology of operators. The method of proof is substantially different from that of Lorch.

A bounded operator E , acting in a Banach space X , is a *projection* if $E^2 = E$. Letting I denote the identity projection, $E(X)$ and $(I - E)(X)$ are strongly closed manifolds.

The projections in X have a natural order: $E_1 \leq E_2$ if $E_1 E_2 = E_2 E_1 = E_1$. If $\{\alpha\}$ is a directed set, and $\alpha_1 \leq \alpha_2$ implies $E_{\alpha_1} \leq E_{\alpha_2}$, then $\{E_\alpha\}$ is *naturally ordered*.

If Y is a subset of X , $w(Y)$ will denote the weak closure of Y . The *span* of Y , denoted by $\text{sp}(Y)$, is the strongly closed linear manifold generated by Y .

Suppose $\{E_\alpha\}$ is a set of projections in X . A projection E such that $E(X) = \text{sp}(\cup_\alpha E_\alpha(X))$ and $(I - E)(X) = \cap_\alpha (I - E_\alpha)(X)$ is the *supremum* of $\{E_\alpha\}$, and is denoted by $\cup_\alpha E_\alpha$.

If $\{E_\alpha\}$ is naturally ordered, y_x is a weak x -cluster point of $\{E_\alpha\}$ if $y_x \in \cap_\alpha w(\{E_\beta(x) \mid \beta \geq \alpha\})$.

The central result of this note follows:

THEOREM 1. *If $\{E_\alpha\}$ is a naturally ordered, uniformly bounded set of projections in X , then $\lim_\alpha E_\alpha = \cup_\alpha E_\alpha$ in the strong topology of operators if and only if $\{E_\alpha\}$ has a weak x -cluster point for each $x \in X$.*

PROOF. Sufficiency. Pick $x \in X$, and a projection E_{α_0} . Let y_x be a weak x -cluster point of $\{E_\alpha\}$. Put $N_\epsilon(y_x) = \{z \mid |(E_{\alpha_0}^* x^*)(z - y_x)| < \epsilon\}$. By assumption, there is a $\beta \geq \alpha_0$ such that $\epsilon > |E_{\alpha_0}^* x^*(E_\beta(x) - y_x)| = |x^*(E_{\alpha_0}(x) - E_{\alpha_0}(y_x))|$. Thus, $E_{\alpha_0}(x) = E_{\alpha_0}(y_x)$.

Since $y_x \in w(\{E_\alpha(x)\})$, it follows by a theorem of Banach [2, p. 58] that $y_x = \lim_{n \rightarrow \infty} T_n(x)$, where $T_n = \sum_{k=1}^n c_k E_{\alpha_k}$.

Clearly, $\lim_\alpha E_\alpha T_n = T_n$ in the strong topology of operators. Furthermore,

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$$\begin{aligned} \|E_\alpha(x) - (E_\alpha T_n)(x)\| &= \|E_\alpha(y_x) - (E_\alpha T_n)(x)\| \\ &\leq M \|y_x - T_n(x)\|, \end{aligned}$$

where $M = \sup_\alpha \|E_\alpha\|$. Thus, $\lim_{n \rightarrow \infty} (E_\alpha T_n)(x) = E_\alpha(x)$, uniformly in α .

By the E. H. Moore theorem on the interchange of limits [4, p. 116],

$$\begin{aligned} y_x &= \lim_{n \rightarrow \infty} \lim_\alpha (E_\alpha T_n)(x) = \lim_\alpha \lim_{n \rightarrow \infty} (E_\alpha T_n)(x) \\ &= \lim_\alpha E_\alpha(x). \end{aligned}$$

Put $E(x) = y_x$. It is now easily verified that E is a projection in X .

If $x \in \bigcap_\alpha (I - E_\alpha)(X)$, then $x = \lim_\alpha (I - E_\alpha)(x) = (I - E)(x)$. Thus, $(I - E)(X) \supset \bigcap_\alpha (I - E_\alpha)(X)$. Since $E(x) = \lim_\alpha E_\alpha(x)$, $E(X) \subset \text{sp}(\bigcup_\alpha E_\alpha(X))$. However, noting $EE_\alpha = E_\alpha$, and $(I - E_\alpha)(I - E) = I - E$, it is clear that $E(X) \supset \text{sp}(\bigcup_\alpha E_\alpha(X))$, and $(I - E)(X) \subset \bigcap_\alpha (I - E_\alpha)(X)$. Therefore, $E = \bigcup_\alpha E_\alpha$.

Necessity is clear.

COROLLARY 1. *If $\{E_\alpha\}$ is a naturally ordered, uniformly bounded set of projections in X , and $\{E_\alpha(x)\}$ is weakly conditionally compact for each $x \in X$, then $\lim_\alpha E_\alpha = \bigcup_\alpha E_\alpha$ in the strong topology of operators.*

PROOF. Pick $x \in X$. If $\beta_0 \geq \alpha_1, \dots, \alpha_n$, then

$$E_{\beta_0}(x) \in \bigcup_{k=1}^n w(\{E_\beta(x) \mid \beta \geq \alpha_k\}).$$

Consequently, $\bigcap_\alpha w(\{E_\beta(x) \mid \beta \geq \alpha\}) \neq \emptyset$.

COROLLARY 2. *If $\{E_\alpha\}$ is a naturally ordered, uniformly bounded set of projections in a reflexive Banach space, then $\lim_\alpha E_\alpha = \bigcup_\alpha E_\alpha$ in the strong topology of operators.*

PROOF. Let $M = \sup_\alpha \|E_\alpha\|$, and $S(x) = \{z \mid \|z\| \leq M\|x\|\}$. By a theorem of Alaoglu [1, p. 255], $S(x)$ is weakly compact. The desired result now follows from Corollary 1.

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