ON THE CONVERGENCE OF ORDERED SETS OF PROJECTIONS[1]

JOHN Y. BARRY

E. R. Lorch has shown [**3**, p. 223] that a uniformly bounded, naturally ordered sequence of projections in a reflexive Banach space converges to its supremum in the strong topology of operators. In this note it is shown that a uniformly bounded, naturally ordered set of projections in a Banach space X having a weak x-cluster point for each $x \in X$ converges to its supremum in the strong topology of operators. The method of proof is substantially different from that of Lorch.

A bounded operator E, acting in a Banach space X, is a *projection* if $E^2 = E$. Letting I denote the identity projection, $E(X)$ and $(I-E)(X)$ are strongly closed manifolds.

The projections in X have a natural order: $E_1 \leqq E_2$ if $E_1E_2 = E_2E_1 = E_1$. If $\{\alpha\}$ is a directed set, and $\alpha_1 \leqq \alpha_2$ implies $E_{\alpha_1} \leqq E_{\alpha_2}$, then $\{E_\alpha\}$ is *naturally ordered.*

If Y is a subset of X, $w(Y)$ will denote the weak closure of Y. The *span of* Y, denoted by $\mathrm{sp}(Y)$, is the strongly closed linear manifold generated by Y.

Suppose $\{E_\alpha\}$ is a set of projections in X. A projection E such that $E(X) = \mathrm{sp}(\bigcup_\alpha E_\alpha(X))$ and $(I-E)(X) = \bigcap_\alpha (I-E_\alpha)(X)$ is the *supremum* of $\{E_\alpha\}$, and is denoted by $\bigcup_\alpha E_\alpha$.

If $\{E_\alpha\}$ is naturally ordered, y_x is a weak *x-cluster point* of $\{E_\alpha\}$ if $y_x \in \bigcap_\alpha w(\{E_\beta(x) \mid \beta \geqq \alpha\})$.

The central result of this note follows:

THEOREM 1. *If $\{E_\alpha\}$ is a naturally ordered, uniformly bounded set of projections in X, then $\lim_\alpha E_\alpha = \bigcup_\alpha E_\alpha$ in the strong topology of operators if and only if $\{E_\alpha\}$ has a weak x-cluster point for each $x \in X$.*

PROOF. Sufficiency. Pick $x \in X$, and a projection E_{α_0}. Let y_x be a weak x-cluster point of $\{E_\alpha\}$. Put $N_\epsilon(y_x) = \{z \mid |(E^*_{\alpha_0}x^*)(z-y_x)| < \epsilon\}$. By assumption, there is a $\beta \geqq \alpha_0$ such that $\epsilon > |E^*_{\alpha_0}x^*(E_\beta(x) - y_x)| = |x^*(E_{\alpha_0}(x) - E_{\alpha_0}(y_x))|$. Thus, $E_{\alpha_0}(x) = E_{\alpha_0}(y_x)$.

Since $y_x \in w(\{E_\alpha(x)\})$, it follows by a theorem of Banach [**2**, p. 58] that $y_x = \lim_{n\to\infty} T_n(x)$, where $T_n = \sum_{k=1}^n c_k^n E_{\alpha_k}$.

Clearly, $\lim_\alpha E_\alpha T_n = T_n$ in the strong topology of operators. Furthermore,

Received by the editors August 3, 1953.

[1] Work sponsored by Office of Naval Research under contract ONR 609 (04).

$$\|E_\alpha(x) - (E_\alpha T_n)(x)\| = \|E_\alpha(y_x) - (E_\alpha T_n)(x)\| \leqq M\|y_x - T_n(x)\|,$$

where $M = \sup_\alpha \|E_\alpha\|$. Thus, $\lim_{n\to\infty} (E_\alpha T_n)(x) = E_\alpha(x)$, uniformly in α.

By the E. H. Moore theorem on the interchange of limits [**4**, p. 116],

$$y_x = \lim_{n\to\infty} \lim_\alpha (E_\alpha T_n(x) = \lim_\alpha \lim_{n\to\infty} (E_\alpha T_n)(x) = \lim_\alpha E_\alpha(x).$$

Put $E(x) = y_x$. It is now easily verified that E is a projection in X.

If $x \in \bigcap_\alpha (I - E_\alpha)(X)$, then $x = \lim_\alpha (I - E_\alpha)(x) = (I - E)(x)$. Thus, $(I-E)(X) \supset \bigcap_\alpha (I-E_\alpha)(X)$. Since $E(x) = \lim_\alpha E_\alpha(x)$, $E(X) \subset \mathrm{sp}(\bigcup_\alpha E_\alpha(X))$. However, noting $EE_\alpha = E_\alpha$, and $(I-E_\alpha)(I-E) = I-E$, it is clear that $E(X) \supset \mathrm{sp}(\bigcup_\alpha E_\alpha(X))$, and $(I-E)(X) \subset \bigcap_\alpha (I-E_\alpha)(X)$. Therefore, $E = \bigcup_\alpha E_\alpha$.

Necessity is clear.

COROLLARY 1. *If* $\{E_\alpha\}$ *is a naturally ordered, uniformly bounded set of projections in* X, *and* $\{E_\alpha(x)\}$ *is weakly conditionally compact for each* $x \in X$, *then* $\lim_\alpha E_\alpha = \bigcup_\alpha E_\alpha$ *in the strong topology of operators.*

PROOF. Pick $x \in X$. If $\beta_0 \geqq \alpha_1, \cdots, \alpha_n$, then

$$E_{\beta_0}(x) \in \bigcup_{k=1}^{n} w(\{E_\beta(x) \mid \beta \geqq \alpha_k\}).$$

Consequently, $\bigcap_\alpha w(\{E_\beta(x) \mid \beta \geqq \alpha\}) \neq \emptyset$.

COROLLARY 2. *If* $\{E_\alpha\}$ *is a naturally ordered, uniformly bounded set of projections in a reflexive Banach space, then* $\lim_\alpha E_\alpha = \bigcup_\alpha E_\alpha$ *in the strong topology of operators.*

PROOF. Let $M = \sup_\alpha \|E_\alpha\|$, and $S(x) = \{z \mid \|z\| \leqq M\|x\|\}$. By a theorem of Alaoglu [**1**, p. 255], $S(x)$ is weakly compact. The desired result now follows from Corollary 1.

REFERENCES

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4. E. H. Moore and H. L. Smith, *A general theory of limits*, Amer. J. Math. vol. 44 (1922) pp. 102–121.

YALE UNIVERSITY