ON PARACOMPACT SPACES

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1. Stone has proved that a space is fully normal $T_1$ if and only if it is paracompact $T_2$. If throughout his proof $T_1$ is deleted and $T_2$ is replaced by $T_4$ (normality), we obtain that a space is fully normal if and only if it is paracompact $T_4$. In this note we prove that $T_2$ can also be replaced by any one of the following:

$T'$. Every two points with disjoint closures have disjoint neighborhoods.

$T'_2$. For each point $x$ and neighborhood $U$ of $x$ there is a neighborhood $V$ of $x$ whose closure is contained in $U$.

$LT_4$. Every point has a neighborhood whose closure is normal.

Moreover, we shall also study a natural decomposition of a space with certain properties and prove that a space is paracompact $T_4$ if and only if it has a retract which is paracompact $T_2$ and meets every non-null closed set.

The following terminology will be used.

**Property *.** If $x$, $y$, $z$ are any three points such that $x \cap y \neq \emptyset$ and $x \cap z \neq \emptyset$, then $y \cap z \neq \emptyset$.

**Property **. Every point has a compact closure.

We remark that

1. $T_2 \rightarrow T_2$ and $T_3 \rightarrow T_3$.

2. $T_4 \rightarrow LT_4 \rightarrow T'_3 \rightarrow T'_2 \rightarrow \text{property *}$.

2. Theorem 1. In a paracompact space, $T'_2$, $T'_3$, $LT_4$ and $T_4$ are equivalent.

**Proof.** It is immediate that $T_4$ implies $T'_2$, $T'_3$, and $LT_4$. To prove that $T'_2$ plus paracompactness implies $T'_3$ and that $T'_3$ plus paracompactness implies $T_4$ we can simply follow Dieudonné's proof with $T_2$ and $T_3$ replaced by $T'_2$ and $T'_3$ respectively. For any paracompact $LT_4$ space there is a locally finite open covering, say $\{ U_a \}$, such that each $\overline{U_a}$ is normal. Since $\{ U_a \}$ is still locally finite, the un-
proved part of Theorem 1 follows from the following more general result.

**Lemma.** A space which is covered by a locally finite system of normal closed sets is normal.

**Proof.** Let $E$, $F$ be disjoint closed sets in a topological space $X$ covered by a locally finite system of closed normal sets $A_a$. For each $\alpha$, there exist, by the normality of $A_a$, open sets $U_a$, $V_a$ in $X$ such that

$$E \cap A_\alpha \subseteq U_\alpha, \quad F \cap A_\alpha \subseteq V_\alpha, \text{ and } U_\alpha \cap V_\alpha \cap A_\alpha = \emptyset.$$  

Now we define

$$P(x) = X - \bigcup \{A_\alpha : A_\alpha \ni x \}, \quad x \in X;$$
$$Q(x) = \bigcap \{U_\alpha : A_\alpha \ni x \}, \quad x \in E;$$
$$R(x) = \bigcap \{V_\alpha : A_\alpha \ni x \}, \quad x \in F.$$  

It follows by the local finiteness of the system $\{A_\alpha\}$ that $P(x)$, $Q(x)$, and $R(x)$ are open. Let

$$U = \bigcup \{P(x) \cap Q(x) : x \in E\}, \quad V = \bigcup \{P(x) \cap R(x) : x \in F\}.$$  

Then $U$, $V$ are disjoint neighborhoods of $E$, $F$.

**Remark.** We can show by examples that in a pointwise paracompact space, $T_\mathcal{I}'$, $T_\mathcal{I}$, $LT_\mathcal{I}$, and $T_\mathcal{I}$ are not equivalent to one another.

**Corollary.** The product of a paracompact $T_4$ space and a compact $T_4$ space is paracompact $T_4$.

To prove this corollary we have only to observe that the product of a paracompact space and a compact space is paracompact\footnote{This is suggested by the referee as a known but unpublished result.} and that the product of two $T_2'$ spaces is $T_2'$.

3. Given any space $X$ with property $\ast$ we can define an equivalence relation such that two points $x$, $y$ of $X$ are equivalent if and only if $x \cap y \neq \emptyset$. This relation yields a decomposition $D$ of $X$, that is, a system of sets, pairwise disjoint, whose union is $X$ and such that two points of $X$ are contained in a same member of $D$ if and only if they are equivalent. Denote by $\phi$ the projection of $X$ onto $D$, i.e., the function of $X$ into $D$ such that $f(x) = \phi$ if $x \in \phi$. There is a topology on $D$ such that a subset $G$ of $D$ is open if and only if $\phi^{-1}(G)$ is open. $D$ with this topology is the natural quotient space of $X$. If $X$ is $T_4$ and then

\footnote{Cf. J. Dieudonné, loc. cit.}
has property *, the natural quotient space of $X$ agrees with one constructed by Čech.\textsuperscript{7} Throughout the rest of this note, $D$ and $\phi$ always denote the natural quotient space and the projection for space $X$ provided $X$ has property *.

**Lemma.** Let $X$ be a space with property * and property **.

(3) Whenever $p \in D$, $\phi^{-1}(p)$ contains a smallest non-null closed set $F_p$. $F_p = \bar{x}$ for $x \in F_p$.

(4) Every open set which meets $F_p$ contains $\phi^{-1}(p)$. Hence for any $x \in X$, if $\bar{x}$ is covered by a system of open sets, $\bar{x}$ is contained in one of them.

(5) If $E$, $F$ are disjoint closed sets in $X$, then $\phi(E) \cap \phi(F) = \emptyset$.

**Proof.** Fix $p \in D$ and let $x_0 \in \phi^{-1}(p)$. Clearly $\{ \bar{x} \cap x_0 : x \in \phi^{-1}(p) \}$ is a system of closed sets in $\bar{x}_0$ and it has, by property *, the finite intersection property. It follows by property ** that 

$$F_p = \cap \{ x : x \in \phi^{-1}(p) \} = \cap \{ \bar{x} \cap \bar{x}_0 : x \in \phi^{-1}(p) \}$$

is a non-null closed set contained in $\phi^{-1}(p)$. For any non-null closed set $F$ contained in $\phi^{-1}(p)$, we have $F \supseteq \bar{x} \supseteq F_p$ whenever $x \in F$. Hence $F_p$ is the smallest. If $x \in F_p$, then $\bar{x} \subseteq F_p \subseteq \bar{x}$. Hence (3) is proved.

By construction $F_p \subseteq \bar{x}$ for each $x \in \phi^{-1}(p)$. Hence, if $U$ is open and meets $F_p$, $U$ meets $\{ x \}$, i.e., $U$ contains each $x \in \phi^{-1}(p)$, proving that $\phi^{-1}(p) \subseteq U$. If $x \in X$ and $\bar{x}$ is covered by a system $U$ of open sets, then some open set of $U$ meets $F_p$ with $p = \phi(x)$, and hence contains $\bar{x}$. This proves (4).

If $E$ and $F$ are disjoint closed sets in $X$, then for $x \in E$ and $y \in F$, $\bar{x} \cap y \subseteq E \cap F = \emptyset$ and so $\phi(x) \neq \phi(y)$. Hence $\phi(E) \cap \phi(F) = \emptyset$, proving (5).

**Lemma.** Let $X$ be a $T_\delta$ space with property **.

(6) $\phi$ is closed.

(7) Given any open covering $\{ U_a \}$ of $X$, $\{ D - \phi(X - U_a) \}$ is an open covering of $D$.

(8) $D$ is $T_3$ (i.e., $T_2$ and $T_3$).

(9) $D$ is $T_3$ (i.e., $T_2$ and $T_3$) or $LT_3$ (i.e., $T_2$ and $LT_3$) according as $X$ is $T_4$ or $LT_4$.

**Proof.** Let $F$ be a closed subset of $X$. Given any point $p$ of $D - \phi(F)$ there is, by (3), a smallest non-null closed set $F_p$ contained in $\phi^{-1}(p)$. Take a point $p_x$ of $F_p$; there is, by $T_\delta'$, a neighborhood $V_p$ of $x_p$ whose closure is contained in $X - F$. Therefore, by (4) and (5), $\phi^{-1}(p) \subseteq V_p \subseteq \bar{V}_p \subseteq X - \phi^{-1}(D - \phi(F))$. Hence $\phi^{-1}(D - \phi(F))$


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= \bigcup \{ V_p : p \in D - \phi(F) \} is open and so \( \phi(F) \) is closed. This proves (6).

Given any open covering \( \{ U_a \} \) of \( X \), \( \{ D - \phi(X - U_a) \} \) is a system of open sets in \( D \) by (6). For each \( p \in D \) there is, by (4), some \( \alpha \) such that \( \phi^{-1}(\alpha) \subset U_a \) and so \( p \in D - \phi(X - U_a) \). Hence (7) is proved.

By (6), \( D \) is a \( T_1 \) space. Therefore \( D \) is \( T_{22} \) if we can show that for a point \( p \) of \( D \) and a neighborhood \( G \) of \( p \) there is a neighborhood of \( p \) whose closure is contained in \( G \). Take a point \( x \) of \( F_p \) and let \( U \) be a neighborhood of \( x \) with \( \overline{U} \subset \phi^{-1}(G) \). Then \( D - \phi(X - U) \) is a neighborhood of \( p \) by (4) and (6) and its closure is contained in \( G \) by (5).

Suppose that \( X \) is \( T_4 \). Given any two disjoint closed subsets \( P, Q \) of \( D, \phi^{-1}(P) \) and \( \phi^{-1}(Q) \) are disjoint closed subsets of \( X \) and they have disjoint neighborhoods \( U \) and \( V \). Applying (6), we can easily see that \( D - \phi(X - U) \) and \( D - \phi(X - V) \) are disjoint neighborhoods of \( P \) and \( Q \), proving that \( D \) is \( T_4 \) and hence \( T_{24} \).

Suppose now that \( X \) is \( LT_4 \). Let \( p \in D \) and \( x \in F_p \); there is, by hypothesis, a neighborhood \( U \) of \( x \) whose closure is normal. By (4) and (6), \( G = D - \phi(X - U) \) is a neighborhood of \( p \). Then there is, by \( T_3 \), a neighborhood \( V \) of \( p \) such that \( \overline{V} \subset G \). Since \( \phi^{-1}(\overline{V}) \subset \overline{U} \) is normal, it follows by the preceding result that \( \overline{V} \) is normal, proving that \( D \) is \( LT_4 \) and hence \( LT_{24} \).

**Lemma.** Let \( X \) be a pointwise paracompact (paracompact) space.

(10) \( X \) has property **.

(11) If \( X \) is \( T_3' \), then \( D \) is pointwise paracompact (paracompact).

**Proof.** Fix a point \( x_0 \) of \( X \). Given any system \( \mathcal{U} \) of open sets whose union contains \( x_0 \), \( \mathcal{U} \cup \{ X - x_0 \} \) is an open covering of \( X \) and it admits a point-finite refinement \( \mathcal{B} \). Let \( \mathcal{B}' = \{ V : V \in \mathcal{B}, \ V \cap x_0 \neq \emptyset \} = \{ V : V \in \mathcal{B}, \ V \ni x_0 \} \) and for each \( V \in \mathcal{B}' \) we take a \( U_V \in \mathcal{U} \) such that \( V \subset U_V \). Then \( \mathcal{U}' = \{ U_V : V \in \mathcal{B}' \} \) is a finite subsystem of \( \mathcal{U} \) which covers \( x_0 \), proving the compactness of \( x_0 \). Hence \( X \) has property **.

Given any open covering \( \{ G_a \} \) of \( D \), \( \{ \phi^{-1}(G_a) \} \) is an open covering of \( X \) and it admits, by hypothesis, a point-finite (locally finite) refinement \( \{ U_\beta \} \). By (7), \( \{ D - \phi(X - U_\beta) \} \) is an open covering of \( D \) which is obviously point-finite (locally finite) and refines \( \{ G_a \} \). Hence \( D \) is pointwise paracompact (paracompact).

**Theorem 2.** A space \( X \) is pointwise paracompact (paracompact) \( T_3' \) if and only if it has a pointwise paracompact (paracompact) \( T_{22} \) retract \( A \) which meets every non-null closed subset of \( X \). Moreover, such a retract \( A \) is \( LT_{24} \) or \( T_{24} \) if and only if \( X \) is \( LT_4 \) or \( T_4 \). Finally, \( A \) is unique up to a homeomorphism and the related retraction is uniquely determined.
Proof. Suppose that $X$ is pointwise paracompact (paracompact) $T'$4. By (2), and (10), $X$ has property * and property **; it follows by (3) that for each $p \in D$ there is a smallest non-null closed set $F_p$ contained in $\phi^{-1}(p)$. We take, for each $p \in D$, a point $x_p$ of $F_p$ and denote by $A$ the set of these points $x_p$. Clearly $A$ meets every non-null closed subset of $X$ and $\phi$ defines a 1-1 mapping $\psi$ of $A$ onto $D$. For any closed subset $F$ of $X$, we have $\psi(F \cap A) = \phi(F)$ which is closed by (6). Hence $\psi$ is a homeomorphism. From this result, we obtain a retraction $f = \psi^{-1}\phi$ of $X$ onto $A$. Moreover, it follows by (8) and (9) that $A$ is $T_{23}$ and that $A$ is $T_{24}$ or $LT_4$ according as $X$ is $T_4$ or $LT_4$.

Conversely suppose that $X$ has a pointwise paracompact (paracompact) $T_{23}$ retract $A$ which meets every non-null closed subset of $X$. Let $f$ be a retraction of $X$ into $A$. For any $x \in X$, $\hat{x} \cap A$ is non-null and $T_2$; therefore it contains exactly one point. From this result, it is easily seen that for any two points $x$, $y$ of $X$, $\hat{x} \cap \hat{y} \neq \emptyset$ if and only if $\hat{x} \cap A = \hat{y} \cap A$. Hence property * as well as property ** holds for $X$, and for each $p \in D$, $\phi^{-1}(p) \cap A$ contains exactly one point contained in $F_p$. From the latter result, we have, for $x \in X$, $f(x) = f(\hat{x}) = \hat{x} \cap A$. Let $x \in X$ and let $U$ be a neighborhood of $\hat{x}$. Then there is a neighborhood $G$ of $\hat{x} \cap A$ in $A$ such that $\hat{x} \cap A \subset G \subset \overline{G} \subset A \subset U \cap A$. Hence $f^{-1}(G)$ is a neighborhood of $x$ whose closure is contained in $U$. This proves that $X$ is $T'_4$.

If $A$ is $T_4$, then for any two disjoint closed subsets $E$, $F$ of $X$, $E \cap A$ and $F \cap A$ have disjoint neighborhoods $G$ and $H$ in $A$ and so $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint neighborhoods of $E$ and $F$. Hence $X$ is also $T_4$. If $A$ is $LT_4$, then for any $x \in X$, $f(x)$ has a neighborhood $G$ in $A$ whose closure in $A$ is normal. It follows by the preceding result that $f^{-1}(G)$ is a neighborhood of $x$ whose closure is normal. Hence $X$ is also $LT_4$.

Given any open covering $\{ U_a \}$ of $X$, $\{ U_a \cap A \}$ is an open covering of $A$ and it admits, by hypothesis, a point-finite (locally finite) refinement $\{ G_\beta \}$. We can easily see that $\{ f^{-1}(G_\beta) \}$ is a point-finite (locally finite) refinement of $\{ U_a \}$. Hence $X$ is pointwise paracompact (paracompact).

According to the above argument, $A$ is homeomorphic to $D$ and hence is unique up to a homeomorphism. Moreover, for any $x \in X$, $\hat{x} \cap A = f(x)$. Hence the retraction $f$ is uniquely determined.

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