ON PARACOMPACT SPACES[1]

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1. Stone[2] has proved that a space is fully normal T_1 if and only if it is paracompact T_2. If throughout his proof T_1 is deleted and T_2 is replaced by T_4 (normality), we obtain that *a space is fully normal if and only if it is paracompact* T_4. In this note we prove that T_2 can also be replaced by any one of the following:

T_2'. Every two points with disjoint closures have disjoint neighborhoods.

T_3'. For each point x and neighborhood U of $\bar{x}$ there is a neighborhood V of x whose closure is contained in U.

LT_4. Every point has a neighborhood whose closure is normal.

Moreover, we shall also study a natural decomposition of a space with certain properties and prove that a space is paracompact T_4 if and only if it has a retract which is paracompact T_2 and meets every non-null closed set.

The following terminology will be used.

PROPERTY *. If x, y, z are any three points such that $\bar{x}\cap\bar{y}\neq\emptyset$ and $\bar{x}\cap\bar{z}\neq\emptyset$, then $\bar{y}\cap\bar{z}\neq\emptyset$.

PROPERTY **. Every point has a compact closure.

We remark that

$$T_2 \to T_2' \quad \text{and} \quad T_3 \to T_3'. \tag{1}$$

$$T_4 \to LT_4 \to T_3' \to T_2' \to \text{property } *. \tag{2}$$

2. THEOREM 1. *In a paracompact space, T_2', T_3', LT_4 and T_4 are equivalent.*

PROOF. It is immediate that T_4 implies T_2', T_3', and LT_4. To prove that T_2' plus paracompactness implies T_3' and that T_3' plus paracompactness implies T_4 we can simply follow Dieudonné's proof[3] with T_2 and T_3 replaced by T_2' and T_3' respectively. For any paracompact LT_4 space there is a locally finite open covering, say $\{U_\alpha\}$, such that each $\overline{U}_\alpha$ is normal. Since $\{\overline{U}_\alpha\}$ is still locally finite, the un-

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[2] Cf. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 977–982.

[3] Cf. J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures Appl. (9) vol. 23 (1944) pp. 65–76.

proved part of Theorem 1 follows from the following more general result.

LEMMA.[4] *A space which is covered by a locally finite system of normal closed sets is normal.*

PROOF. Let E, F be disjoint closed sets in a topological space X covered by a locally finite system of closed normal sets A_α. For each α, there exist, by the normality of A_α, open sets U_α, V_α in X such that

$$E \cap A_\alpha \subset U_\alpha, \quad F \cap A_\alpha \subset V_\alpha, \text{ and } U_\alpha \cap V_\alpha \cap A_\alpha = \emptyset.$$

Now we define

$$\begin{aligned} P(x) &= X - \bigcup\{A_\alpha : A_\alpha \not\ni x\}, && x \in X; \\ Q(x) &= \bigcap\{U_\alpha : A_\alpha \ni x\}, && x \in E; \\ R(x) &= \bigcap\{V_\alpha : A_\alpha \ni x\}, && x \in F. \end{aligned}$$

It follows by the local finiteness of the system $\{A_\alpha\}$ that $P(x)$, $Q(x)$, and $R(x)$ are open. Let

$$U = \bigcup\{P(x) \cap Q(x) : x \in E\}, \qquad V = \bigcup\{P(x) \cap R(x) : x \in F\}.$$

Then U, V are disjoint neighborhoods of E, F.

REMARK. We can show by examples that in a pointwise paracompact space, T_2', T_3', LT_4, and T_4 are not equivalent to one another.

COROLLARY.[5] *The product of a paracompact T_4 space and a compact T_4 space is paracompact T_4.*

To prove this corollary we have only to observe that the product of a paracompact space and a compact space is paracompact[6] and that the product of two T_2' spaces is T_2'.

3. Given any space X with property * we can define an equivalence relation such that two points x, y of X are equivalent if and only if $\bar{x} \cap \bar{y} \neq \emptyset$. This relation yields a decomposition D of X, that is, a system of sets, pairwise disjoint, whose union is X and such that two points of X are contained in a same member of D if and only if they are equivalent. Denote by ϕ the *projection* of X onto D, i.e., the function of X into D such that $f(x) = p$ if $x \in p$. There is a topology on D such that a subset G of D is open if and only if $\phi^{-1}(G)$ is open. D with this topology is the *natural quotient space* of X. If X is T_4 and then

[4] This is suggested by the referee as a known but unpublished result.

[5] Cf. H. Miyazaki, *A note on paracompact spaces*, Tôhoku Math. J. (2) vol. 4 (1952) pp. 88–92.

[6] Cf. J. Dieudonné, loc. cit.

has property *, the natural quotient space of X agrees with one constructed by Čech.[7] Throughout the rest of this note, D and ϕ always denote the natural quotient space and the projection for space X provided X has property *.

LEMMA. *Let X be a space with property * and property **.*

(3) *Whenever $p \in D$, $\phi^{-1}(p)$ contains a smallest non-null closed set F_p. $F_p = \bar{x}$ for $x \in F_p$.*

(4) *Every open set which meets F_p contains $\phi^{-1}(p)$. Hence for any $x \in X$, if $\bar{x}$ is covered by a system of open sets, $\bar{x}$ is contained in one of them.*

(5) *If E, F are disjoint closed sets in X, then $\phi(E) \cap \phi(F) = \emptyset$.*

PROOF. Fix $p \in D$ and let $x_0 \in \phi^{-1}(p)$. Clearly $\{\bar{x} \cap \bar{x}_0 : x \in \phi^{-1}(p)\}$ is a system of closed sets in $\bar{x}_0$ and it has, by property *, the finite intersection property. It follows by property ** that

$$F_p = \bigcap \{x : x \in \phi^{-1}(p)\} = \bigcap \{\bar{x} \cap \bar{x}_0 : x \in \phi^{-1}(p)\}$$

is a non-null closed set contained in $\phi^{-1}(p)$. For any non-null closed set F contained in $\phi^{-1}(p)$, we have $F \supset \bar{x} \supset F_p$ whenever $x \in F$. Hence F_p is the smallest. If $x \in F_p$, then $\bar{x} \subset F_p \subset \bar{x}$. Hence (3) is proved.

By construction $F_p \subset \bar{x}$ for each $x \in \phi^{-1}(p)$. Hence, if U is open and meets F_p, U meets $\{x\}$, i.e., U contains each $x \in \phi^{-1}(p)$, proving that $\phi^{-1}(p) \subset U$. If $x \in X$ and $\bar{x}$ is covered by a system $\mathfrak{U}$ of open sets, then some open set of $\mathfrak{U}$ meets F_p with $p = \phi(x)$, and hence contains $\bar{x}$. This proves (4).

If E and F are disjoint closed sets in X, then for $x \in E$ and $y \in F$, $\bar{x} \cap \bar{y} \subset E \cap F = \emptyset$ and so $\phi(x) \neq \phi(y)$. Hence $\phi(E) \cap \phi(F) = \emptyset$, proving (5).

LEMMA. *Let X be a T_3' space with property **.*

(6) *ϕ is closed.*

(7) *Given any open covering $\{U_\alpha\}$ of X, $\{D - \phi(X - U_\alpha)\}$ is an open covering of D.*

(8) *D is T_{23} (i.e., T_2 and T_3).*

(9) *D is T_{24} (i.e., T_2 and T_4) or LT_{24} (i.e., T_2 and LT_4) according as X is T_4 or LT_4.*

PROOF. Let F be a closed subset of X. Given any point p of $D - \phi(F)$ there is, by (3), a smallest non-null closed set F_p contained in $\phi^{-1}(p)$. Take a point x_p of F_p; there is, by T_3', a neighborhood V_p of x_p whose closure is contained in $X - F$. Therefore, by (4) and (5), $\phi^{-1}(p) \subset V_p \subset \overline{V}_p \subset X - \phi^{-1}\phi(F) = \phi^{-1}(D - \phi(F))$. Hence $\phi^{-1}(D - \phi(F))$

[7] Cf. E. Čech, *On bicompact spaces*, Ann. of Math. (2) vol. 38 (1937) pp. 823–844.

$=\bigcup\{V_p: p\in D-\phi(F)\}$ is open and so $\phi(F)$ is closed. This proves (6).

Given any open covering $\{U_\alpha\}$ of X, $\{D-\phi(X-U_\alpha)\}$ is a system of open sets in D by (6). For each $p\in D$ there is, by (4), some α such that $\phi^{-1}(p)\subset U_\alpha$ and so $p\in D-\phi(X-U_\alpha)$. Hence (7) is proved.

By (6), D is a T_1 space. Therefore D is T_{23} if we can show that for a point p of D and a neighborhood G of p there is a neighborhood of p whose closure is contained in G. Take a point x of F_p and let U be a neighborhood of x with $\overline{U}\subset\phi^{-1}(G)$. Then $D-\phi(X-U)$ is a neighborhood of p by (4) and (6) and its closure is contained in G by (5).

Suppose that X is T_4. Given any two disjoint closed subsets P, Q of D, $\phi^{-1}(P)$ and $\phi^{-1}(Q)$ are disjoint closed subsets of X and they have disjoint neighborhoods U and V. Applying (6), we can easily see that $D-\phi(X-U)$ and $D-\phi(X-V)$ are disjoint neighborhoods of P and Q, proving that D is T_4 and hence T_{24}.

Suppose now that X is LT_4. Let $p\in D$ and $x\in F_p$; there is, by hypothesis, a neighborhood U of x whose closure is normal. By (4) and (6), $G=D-\phi(X-U)$ is a neighborhood of p. Then there is, by T_3, a neighborhood V of p such that $\overline{V}\subset G$. Since $\phi^{-1}(\overline{V})\subset\overline{U}$ is normal, it follows by the preceding result that $\overline{V}$ is normal, proving that D is LT_4 and hence LT_{24}.

LEMMA. *Let X be a pointwise paracompact (paracompact) space.*

(10) *X has property* **.

(11) *If X is T_3', then D is pointwise paracompact (paracompact).*

PROOF. Fix a point x_0 of X. Given any system $\mathfrak{U}$ of open sets whose union contains $\bar{x}_0$, $\mathfrak{U}\cup\{X-\bar{x}_0\}$ is an open covering of X and it admits a point-finite refinement $\mathfrak{V}$. Let $\mathfrak{V}'=\{V: V\in\mathfrak{V},\ V\cap\bar{x}_0\neq\varnothing\}=\{V: V\in\mathfrak{V},\ V\ni x_0\}$ and for each $V\in\mathfrak{V}'$ we take a $U_V\in\mathfrak{U}$ such that $V\subset U_V$. Then $\mathfrak{U}'=\{U_V: V\in\mathfrak{V}'\}$ is a finite subsystem of $\mathfrak{U}$ which covers $\bar{x}_0$, proving the compactness of $\bar{x}_0$. Hence X has property **.

Given any open covering $\{G_\alpha\}$ of D, $\{\phi^{-1}(G_\alpha)\}$ is an open covering of X and it admits, by hypothesis, a point-finite (locally finite) refinement $\{U_\beta\}$. By (7), $\{D-\phi(X-U_\beta)\}$ is an open covering of D which is obviously point-finite (locally finite) and refines $\{G_\alpha\}$. Hence D is pointwise paracompact (paracompact).

THEOREM 2. *A space X is pointwise paracompact (paracompact) T_3' if and only if it has a pointwise paracompact (paracompact) T_{23} retract A which meets every non-null closed subset of X. Moreover, such a retract A is LT_{24} or T_{24} if and only if X is LT_4 or T_4. Finally, A is unique up to a homeomorphism and the related retraction is uniquely determined.*

PROOF. Suppose that X is pointwise paracompact (paracompact) T_3'. By (2), and (10), X has property * and property **; it follows by (3) that for each $p \in D$ there is a smallest non-null closed set F_p contained in $\phi^{-1}(p)$. We take, for each $p \in D$, a point x_p of F_p and denote by A the set of these points x_p. Clearly A meets every non-null closed subset of X and ϕ defines a 1-1 mapping ψ of A onto D. For any closed subset F of X, we have $\psi(F \cap A) = \phi(F)$ which is closed by (6). Hence ψ is a homeomorphism. From this result, we obtain a retraction $f = \psi^{-1}\phi$ of X onto A. Moreover, it follows by (8) and (9) that A is T_{23} and that A is T_{24} or LT_4 according as X is T_4 or LT_4.

Conversely suppose that X has a pointwise paracompact (paracompact) T_{23} retract A which meets every non-null closed subset of X. Let f be a retraction of X into A. For any $x \in X$, $\bar{x} \cap A$ is non-null and T_2; therefore it contains exactly one point. From this result, it is easily seen that for any two points x, y of X, $\bar{x} \cap \bar{y} \neq \emptyset$ if and only if $\bar{x} \cap A = \bar{y} \cap A$. Hence property * as well as property ** holds for X, and for each $p \in D$, $\phi^{-1}(p) \cap A$ contains exactly one point contained in F_p. From the latter result, we have, for $x \in X$, $f(x) = f(\bar{x}) = \bar{x} \cap A$. Let $x \in X$ and let U be a neighborhood of $\bar{x}$. Then there is a neighborhood G of $\bar{x} \cap A$ in A such that $\bar{x} \cap A \subset G \subset \bar{G} \cap A \subset U \cap A$. Hence $f^{-1}(G)$ is a neighborhood of x whose closure is contained in U. This proves that X is T_3'.

If A is T_4, then for any two disjoint closed subsets E, F of X, $E \cap A$ and $F \cap A$ have disjoint neighborhoods G and H in A and so $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint neighborhoods of E and F. Hence X is also T_4. If A is LT_4, then for any $x \in X$, $f(x)$ has a neighborhood G in A whose closure in A is normal. It follows by the preceding result that $f^{-1}(G)$ is a neighborhood of x whose closure is normal. Hence X is also LT_4.

Given any open covering $\{U_\alpha\}$ of X, $\{U_\alpha \cap A\}$ is an open covering of A and it admits, by hypothesis, a point-finite (locally finite) refinement $\{G_\beta\}$. We can easily see that $\{f^{-1}(G_\beta)\}$ is a point-finite (locally finite) refinement of $\{U_\alpha\}$. Hence X is pointwise paracompact (paracompact).

According to the above argument, A is homeomorphic to D and hence is unique up to a homeomorphism. Moreover, for any $x \in X$, $\bar{x} \cap A = f(x)$. Hence the retraction f is uniquely determined.

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