In studying a general semigroup \(S\), a natural thing to do is to decompose \(S\) (if possible) into the class sum of a set \(\{S_\alpha; \alpha \in I\}\) of mutually disjoint subsemigroups \(S_\alpha\) such that (1) each \(S_\alpha\) belongs to some more or less restrictive type \(\mathfrak{T}\) of semigroup, and (2) the product \(S_\alpha S_\beta\) of any two of them is wholly contained in a third: \(S_\alpha S_\beta \subseteq S_\gamma\), for some \(\gamma \in I\) depending upon \(\alpha\) and \(\beta\). We shall then say that \(S\) is a band of semigroups of type \(\mathfrak{T}\). If, for every \(\alpha\) and \(\beta\) in \(I\), \(S_\alpha S_\beta \) and \(S_\beta S_\alpha\) are both contained in the same \(S_\gamma\), then we shall call \(S\) a semilattice of semigroups of type \(\mathfrak{T}\).

We shall also be concerned with the following specialization of the notion of band of semigroups. Suppose that \(I\) is the direct product \(J \times K\) of two classes \(J\) and \(K\). The subsemigroups \(S_\alpha\) are then described by two subscripts: \(S_{i\kappa}\) \((i \in J, \kappa \in K)\). Suppose moreover that \(S_{i\kappa} S_{j\lambda} \subseteq S_{\alpha\lambda}\) for all \(i, j \in J\) and all \(\kappa, \lambda \in K\). We shall then call \(S\) a matrix of semigroups of type \(\mathfrak{T}\). The primary purpose of the present paper is to show (Theorem 4) that a band of semigroups of type \(\mathfrak{T}\) is a semilattice of semigroups each of which is a matrix of semigroups of type \(\mathfrak{T}\).

The rest of the paper is devoted to giving necessary and sufficient conditions on a semigroup \(S\) that it be a band or a semilattice of (1) simple semigroups, (2) completely simple semigroups, and (3) groups. (Throughout this paper we use the term simple to mean simple without zero, i.e. a simple semigroup is one containing no proper two-sided ideal whatever.) For (1), we have the elegant condition, \(a \in S a^2 S\) for all \(a \in S\), due to Olaf Andersen [1]. If a semigroup \(S\) is a class sum of [completely] simple semigroups, it is also a semilattice of [completely] simple semigroups. But a class sum of groups need not be a band of groups, nor need a band of groups be a semilattice of groups; these three categories are characterized by Theorems 6, 7, and 8, respectively.

We note that a semigroup \(S\) is a “band of groups of order one” if and only if each element of \(S\) is idempotent. In this case we call \(S\) simply a “band,” and consequently make the definition: a band is a semigroup every element of which is idempotent. By the same token, we define a semilattice to be a commutative band. A “matrix of groups
of order one" will be called a “rectangular band.” Thus we define a
rectangular band to be the direct product $J \times K$ of two classes $J$ and
$K$, with multiplication defined by

$$ (i, \kappa)(j, \lambda) = (i, \lambda) \quad (\text{all } i, j \in J; \kappa, \lambda \in K). $$

The reason for the term is the following. Think of $J \times K$ as a rectangu-
lar array of points, the point $\alpha = (i, \kappa)$ lying in the $i$th row and $\kappa$th
column. Then $\alpha = (i, \kappa)$ and $\beta = (j, \lambda)$ are opposite vertices of a
rectangle of which the other two vertices are $\alpha \beta = (i, \lambda)$ and $\beta \alpha = (j, \kappa)$.

**Theorem 1.** A semigroup $S$ is a band of semigroups of type $\mathfrak{C}$ if and
only if there exists a band $I$ and a homomorphism $\phi$ of $S$ onto $I$ such that
the inverse image $\phi^{-1}(\alpha)$ of each element $\alpha$ of $I$ is a subsemigroup $S_\alpha$ of
$S$ of type $\mathfrak{C}$.

**Proof.** Let $S$ be a band of semigroups $S_\alpha$ ($\alpha \in I$) of type $\mathfrak{C}$. Then
to each pair of elements $\alpha, \beta$ of the index class $I$ corresponds a unique
element $\gamma$ of $I$ such that $S_\alpha S_\beta \subseteq S_\gamma$. Define $\alpha \beta = \gamma$. Since $S_\alpha S_\beta \cdot S_\gamma = S_\alpha \cdot S_\beta S_\gamma$, and $S_\alpha S_\alpha \subseteq S_\alpha$, this multiplication is associative and
idempotent, i.e., $I$ becomes a band thereunder. Each element $a$ of $S$
belongs to exactly one $S_\alpha$, and, if we define $\phi(a) = \alpha$, the mapping $\phi$ of
$S$ onto $I$ is clearly a homomorphism with the property stated in the
theorem. The converse is evident.

We shall use the expression, “$S$ is a band $I$ of semigroups $S_\alpha$ ($\alpha \in I$)
of type $\mathfrak{C}$,” to indicate the situation described in Theorem 1. $\phi$ will
be called the natural homomorphism of $S$ onto $I$.

We shall call a semigroup $S$ simple if $S_\alpha S = S$ for every $a \in S$, and
completely simple if $S$ is simple and contains a primitive idempotent
(cf. Rees [3]). (An idempotent $e$ is primitive if from $f^2 = f$ and $ef = fe$
we conclude $f = e$.) We shall not have occasion to consider semi-
groups with a zero element, and we consequently omit the phrase
without zero. Rees [2] showed (in particular) that every completely
simple semigroup is isomorphic with a “matrix semigroup $S$ over a
group $G,”$ defined as follows. Let $J$ and $K$ be any two sets. Let $S$ be
the set of all triples $(a; i, \kappa)$, with $a \in G$, $i \in J$, $\kappa \in K$, and let the
product of two elements of $S$ be defined by

$$ (a; i, \kappa)(b; j, \lambda) = (ap_{\kappa j}b; i, \lambda) \quad (a, b \in G; i, j \in J; \kappa, \lambda \in K), $$

where $(p_{\kappa j})$ is a fixed $K \times J$ matrix of elements $p_{\kappa j}$ of $G$. Since Susch-
kekewitsch [4] proved essentially the same theorem for finite $S$, we
shall refer to this result as the Suschkewitsch-Rees Theorem.

The following theorem is simply a restatement, in the terminology of
the present paper, of Theorem 2 of an earlier paper [5].
Theorem 2. A semigroup is a class sum of groups if and only if it is a semilattice of completely simple semigroups.

We remark that if a semigroup $S$ is a class sum of groups, then it is a class sum of mutually disjoint groups. This follows from Theorem 1 of [5], and the evident fact that $S$ admits relative inverses. A similar remark holds if $S$ is a class sum of simple or of completely simple semigroups, the former by Andersen's Theorem (stated below) and the latter by Theorem 2.

We next express in our present terminology a result due to McLean [6]. Call a band nowhere commutative if $ab = ba$ implies $a = b$.

Theorem (McLean). Every band is a semilattice of nowhere commutative bands.

Since a rectangular band is clearly nowhere commutative, the next theorem sharpens that of McLean.

Theorem 3. Every band is a semilattice of rectangular bands.

Proof. A band is a class sum of groups (of order one), and hence, by Theorem 2, is a semilattice of completely simple bands. Applying the Suschkewitsch-Rees Theorem to a completely simple band $B$, we see that the structure group of $B$ must have order one, and hence the law (2) of multiplication reduces to (1).

We come now to the main result of the present paper.

Theorem 4. A band of semigroups of type $\mathfrak{C}$ is a semilattice of semigroups each of which is a matrix of semigroups of type $\mathfrak{C}$.

Proof. Let $S$ be a band $I$ of semigroups $S_{\alpha}$ ($\alpha \in I$) of type $\mathfrak{C}$. By Theorem 3, $I$ is itself a semilattice $P$ of rectangular bands $I_\tau$ ($\tau \in P$). Let $\phi$ be the natural homomorphism of $S$ onto $I$, and let $\psi$ be that of $I$ onto $P$. The mapping $a \rightarrow \psi(\phi(a))$ is clearly a homomorphism of $S$ onto $P$. The inverse image $S'_{\tau}$ of an element $\tau$ of $P$ is the class sum of all the $S_{\alpha}$ for which $\psi(\alpha) = \tau$, i.e. for which $\alpha \in I_\tau$. Describing the elements $a$ of $I$, as pairs $(i, k)$ of elements of index classes $J_i$ and $K_\kappa$, as above, then we may write $S_{i k}$ for $S_{\alpha}$ and obtain $S_{i k} S_{j \lambda} \subseteq S_{i \lambda}$ for all $i, j \in J_\tau$ and all $\kappa, \lambda \in K_\tau$. Thus $S'_{\tau}$ is a matrix of semigroups $S'_{i \kappa}$, and the latter (being the $S_{\alpha}$) are of type $\mathfrak{C}$.


Proof. Let $S$ be a matrix of simple semigroups $S_{i \kappa}$ ($i \in J, \kappa \in K$). Let $a$ and $b$ be arbitrary elements of $S$; we are to show that there
exist \( x, y \) in \( S \) such that \( xay = b \). Suppose \( a \in S_{i\kappa} \) and \( b \in S_{j\lambda} \). Then \( bab \in S_{j\lambda} \). Since \( S_{j\lambda} \) is simple, there exist \( u, v \) in \( S_{j\lambda} \) such that \( u \cdot bab \cdot v = b \). Hence we may take \( x = ub, y = bv \).

Now suppose that the semigroups \( S_{i\kappa} \) are completely simple. To show that \( S \) is completely simple, it suffices to show that any idempotent element \( e \) of \( S_{i\kappa} \) is primitive in \( S \), as well as in \( S_{i\kappa} \). Let \( f \) be an idempotent element of \( S \) such that \( fe = ef = f \). Suppose \( f \in S_{j\lambda} \). Then \( fe \in S_{j\lambda} S_{i\kappa} \subseteq S_{i\kappa} \) and \( ef \in S_{i\kappa} S_{j\lambda} \subseteq S_{i\kappa} \). Hence \( S_{j\lambda} \) and \( S_{j\lambda} \) must be the same as \( S_{j\lambda} \), i.e. \( i = j \) and \( \kappa = \lambda \). Thus \( f \in S_{i\kappa} \), and since \( e \) is primitive in \( S_{i\kappa} \), we conclude that \( f = e \).

From Theorem 5 and the Suschkewitsch-Rees Theorem, it follows that a semigroup is a matrix of groups if and only if it is completely simple.

The following is Theorem 9.5 in [1]. (A proof may be constructed for it by suitably modifying the proof of Theorem 2.)*

**Theorem (Croisot-Andersen).** The following three assertions concerning a semigroup \( S \) are equivalent:

(A) \( S \) is a class sum of simple semigroups.

(B) \( S \) is a semilattice of simple semigroups.

(C) \( a \in S a^2 S \) for every \( a \in S \).

Let us call two elements \( a, b \) of a semigroup \( S \) \( r \)-associate (\( l \)-associate) if they generate the same right (left) ideal of \( S \), and \( r \cap l \)-associate if they are both \( r \)- and \( l \)-associate. The following is Theorem 7 in [7]:

**Theorem (J. A. Green).** If an element \( a \) of a semigroup \( S \) is \( r \cap l \)-associate with \( a^2 \), then the set of all elements of \( S \) which are \( r \cap l \)-associate with \( a \) is a subgroup of \( S \).

**Theorem 6.** The following four assertions concerning a semigroup \( S \) are equivalent:

(A) \( S \) is a class sum of completely simple semigroups.

(B) \( S \) is a class sum of groups.

(C) \( S \) is a semilattice of completely simple semigroups.

(D) \( a \in S a^2 \cap a^2 S \) for every \( a \in S \).

**Proof.** (A) implies (B), since every completely simple semigroup is a class sum of groups. (B) implies (C) by Theorem 2, and evidently (C) implies (A). (B) implies (D), for, if \( a \in S \), then \( a \) and \( a^2 \) both belong to the same subgroup \( G \) of \( S \), so that \( a \in G a^2 \cap a^2 G \subseteq S a^2 \cap a^2 S \). (D) implies (B), for, if \( a \in S \), (D) implies that \( a \) and \( a^2 \) are

* See footnote 1.
$r \cap l$-associate, and hence $a$ belongs to a subgroup of $S$ by Green's theorem.

**Theorem 7.** A semigroup $S$ is a band of groups if and only if it satisfies the following two conditions:

1. $a \in Sa^2 \cap a^2 S$ for every $a \in S$;
2. for every pair of elements $a$, $b$ of $S$, $Sba = Sba^2$ and $abS = a^2 bS$.

**Proof.** Let $S$ be a band of groups. In particular, $S$ is a class sum of groups, and (1) follows as in the proof of Theorem 6. Let $a$, $b \in S$. From the fact that $a$ and $a^2$ belong to the same subgroup of $S$, and the assumption that $S$ is a band of groups, we conclude that $ba$ and $ba^2$ belong to the same subgroup of $S$, and hence $Sba = Sba^2$. Similarly, $abS = a^2 bS$.

Suppose conversely that (1) and (2) hold in $S$. By (1) and Green's Theorem, $S$ is the class sum of groups $G_a (\alpha \in I)$. Let $a$, $b \in S$. Let $G_a$ be the subgroup of $S$ to which $a$ belongs, let $e_a$ be the identity element of $G_a$, and let $a^{-1}$ be the inverse of $a$ in $G_a$. Replacing $b$ by $a^{-1} b$ in the second part of (2), we conclude $e_a b S = ab S$. If $a'$ is any other element of $G_a$, we conclude similarly that $e_a b S = a' b S$, and hence $ab S = a' b S$. Evidently $Sa = Sa'$, and hence $Sab = Sa'b$. Consequently $ab$ and $a'b$ are $r \cap l$-associate. Now, in a semigroup $S$ which is the class sum of groups, two elements of $S$ are $r \cap l$-associate if and only if they belong to the same subgroup of $S$. Consequently $ab$ and $a'b$ belong to the same subgroup of $S$. By the left-right dual of this argument, $ba$ and $ba'$ also belong to the same subgroup of $S$. Thus the relation of belonging to the same subgroup of $S$ is a congruence relation, whence $S$ is a band of groups.

**Theorem 8.** A semigroup $S$ is a semilattice of groups if and only if it satisfies the following two conditions:

1. $a \in Sa^2 \cap a^2 S$ for every $a \in S$;
2. if $e$ and $f$ are idempotent elements of $S$, then $ef = fe$.

**Proof.** Assume that $S$ is a semilattice $I$ of groups $G_\alpha (\alpha \in I)$. Since $S$ is in particular a class sum of groups, (1) follows as in the proof of Theorem 6. To show (2), let $e$ and $f$ be idempotent elements of $S$. Then $e \in G_\alpha$ and $f \in G_\beta$ for some $\alpha, \beta \in I$. Since $G_\alpha$ and $G_\beta$ are groups, $e$ and $f$ must be the identity elements thereof: $e = e_\alpha$, $f = e_\beta$. Let $\gamma = \alpha \beta$ (= $\beta \alpha$ since $I$ is by hypothesis commutative). Let $e_\gamma$ be the identity element of $G_\gamma$. Since $e_\alpha e_\beta \in G_\gamma$ we have $e_\gamma \cdot e_\alpha e_\beta = e_\gamma e_\beta$.

Now $e_\gamma e_a \in G_\gamma a = G_\gamma$, so that $e_\gamma e_a \cdot e_\gamma = e_\gamma e_a$. Hence

$\text{Added in proof.}$ This result has also been noted by R. Croisot [8, p. 375].
Since $e_\gamma$ is the only idempotent element of $G_\gamma$, we conclude that $e_\gamma e_\alpha = e_\gamma$. Similarly, $e_\gamma e_\beta = e_\gamma$. Hence

$$e_\alpha e_\beta = e_\gamma e_\alpha e_\beta = e_\gamma.$$

Thus $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta$ in $I$, whence $e_{\beta e_\alpha} = e_{\beta e_\alpha} = e_{\alpha e_\beta} = e_{\alpha e_\beta}$.

Conversely, let $S$ be a semigroup satisfying conditions (1) and (2). By (1) and Green’s theorem, $S$ is a class sum of groups. From Theorem 3 of [5] and condition (2) it then follows that $S$ is a semilattice of groups; moreover, the theorem cited provides an explicit construction for any such $S$.

References


THE JOHNS HOPKINS UNIVERSITY