BANDS OF SEMIGROUPS

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In studying a general semigroup S, a natural thing to do is to decompose S (if possible) into the class sum of a set $\{S_\alpha;\ \alpha \in I\}$ of mutually disjoint subsemigroups S_α such that (1) each S_α belongs to some more or less restrictive type $\mathfrak{T}$ of semigroup, and (2) the product $S_\alpha S_\beta$ of any two of them is wholly contained in a third: $S_\alpha S_\beta \subseteq S_\gamma$, for some $\gamma \in I$ depending upon α and β. We shall then say that S is a *band of semigroups of type* $\mathfrak{T}$. If, for every α and β in I, $S_\alpha S_\beta$ and $S_\beta S_\alpha$ are both contained in the same S_γ, then we shall call S a *semilattice of semigroups of type* $\mathfrak{T}$.

We shall also be concerned with the following specialization of the notion of band of semigroups. Suppose that I is the direct product $J \times K$ of two classes J and K. The subsemigroups S_α are then described by two subscripts: $S_{i\kappa}$ $(i \in J,\ \kappa \in K)$. Suppose moreover that $S_{i\kappa} S_{j\lambda} \subseteq S_{i\lambda}$ for all $i, j \in J$ and all $\kappa, \lambda \in K$. We shall then call S a *matrix of semigroups of type* $\mathfrak{T}$. The primary purpose of the present paper is to show (Theorem 4) that a band of semigroups of type $\mathfrak{T}$ is a semilattice of semigroups each of which is a matrix of semigroups of type $\mathfrak{T}$.

The rest of the paper is devoted to giving necessary and sufficient conditions on a semigroup S that it be a band or a semilattice of (1) simple semigroups, (2) completely simple semigroups, and (3) groups. (Throughout this paper we use the term *simple* to mean *simple without zero*, i.e. a simple semigroup is one containing no proper two-sided ideal whatever.) For (1), we have the elegant condition, $a \in Sa^2S$ for all $a \in S$, due to[1] Olaf Andersen [1]. If a semigroup S is a class sum of [completely] simple semigroups, it is also a semilattice of [completely] simple semigroups. But a class sum of groups need not be a band of groups, nor need a band of groups be a semilattice of groups; these three categories are characterized by Theorems 6, 7, and 8, respectively.

We note that a semigroup S is a "band of groups of order one" if and only if each element of S is idempotent. In this case we call S simply a "band," and consequently make the definition: a *band* is a semigroup every element of which is idempotent. By the same token, we define a *semilattice* to be a commutative band. A "matrix of groups

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[1] (Added in proof.) This result has also been obtained recently by R. Croisot [8, p. 369].

of order one" will be called a "rectangular band." Thus we define a *rectangular band* to be the direct product $J\times K$ of two classes J and K, with multiplication defined by

$$(1) \qquad (i, \kappa)(j, \lambda) = (i, \lambda) \quad (\text{all } i, j \in J; \kappa, \lambda \in K).$$

The reason for the term is the following. Think of $J\times K$ as a rectangular array of points, the point $\alpha=(i, \kappa)$ lying in the ith row and κth column. Then $\alpha=(i, \kappa)$ and $\beta=(j, \lambda)$ are opposite vertices of a rectangle of which the other two vertices are $\alpha\beta=(i, \lambda)$ and $\beta\alpha=(j, \kappa)$.

THEOREM 1. *A semigroup S is a band of semigroups of type $\mathfrak{T}$ if and only if there exists a band I and a homomorphism ϕ of S onto I such that the inverse image $\phi^{-1}(\alpha)$ of each element α of I is a subsemigroup S_α of S of type $\mathfrak{T}$.*

PROOF. Let S be a band of semigroups S_α $(\alpha\in I)$ of type $\mathfrak{T}$. Then to each pair of elements α, β of the index class I corresponds a unique element γ of I such that $S_\alpha S_\beta\subseteq S_\gamma$. Define $\alpha\beta=\gamma$. Since $S_\alpha S_\beta\cdot S_\gamma=S_\alpha\cdot S_\beta S_\gamma$, and $S_\alpha S_\alpha\subseteq S_\alpha$, this multiplication is associative and idempotent, i.e. I becomes a band thereunder. Each element a of S belongs to exactly one S_α, and, if we define $\phi(a)=\alpha$, the mapping ϕ of S onto I is clearly a homomorphism with the property stated in the theorem. The converse is evident.

We shall use the expression, "S is a band I of semigroups S_α $(\alpha\in I)$ of type $\mathfrak{T}$," to indicate the situation described in Theorem 1. ϕ will be called the natural homomorphism of S onto I.

We shall call a semigroup S *simple* if $SaS=S$ for every $a\in S$, and *completely simple* if S is simple and contains a primitive idempotent (cf. Rees [3]). (An idempotent e is primitive if from $f^2=f$ and $ef=fe=f$ we conclude $f=e$.) We shall not have occasion to consider semigroups with a zero element, and we consequently omit the phrase *without zero*. Rees [2] showed (in particular) that every completely simple semigroup is isomorphic with a "matrix semigroup S over a group G," defined as follows. Let J and K be any two sets. Let S be the set of all triples $(a; i, \kappa)$, with $a\in G$, $i\in J$, $\kappa\in K$, and let the product of two elements of S be defined by

$$(2) \qquad (a; i, \kappa)(b; j, \lambda) = (ap_{\kappa j}b; i, \lambda) \qquad (a, b \in G; i, j \in J; \kappa, \lambda \in K),$$

where $(p_{\kappa j})$ is a fixed $K\times J$ matrix of elements $p_{\kappa j}$ of G. Since Suschkewitsch [4] proved essentially the same theorem for finite S, we shall refer to this result as the Suschkewitsch-Rees Theorem.

The following theorem is simply a restatement, in the terminology of the present paper, of Theorem 2 of an earlier paper [5].

THEOREM 2. *A semigroup is a class sum of groups if and only if it is a semilattice of completely simple semigroups.*

We remark that if a semigroup S is a class sum of groups, then it is a class sum of mutually disjoint groups. This follows from Theorem 1 of [5], and the evident fact that S admits relative inverses. A similar remark holds if S is a class sum of simple or of completely simple semigroups, the former by Andersen's Theorem (stated below) and the latter by Theorem 2.

We next express in our present terminology a result due to McLean [6]. Call a band *nowhere commutative* if $ab=ba$ implies $a=b$.

THEOREM (McLean). *Every band is a semilattice of nowhere commutative bands.*

Since a rectangular band is clearly nowhere commutative, the next theorem sharpens that of McLean.

THEOREM 3. *Every band is a semilattice of rectangular bands.*

PROOF. A band is a class sum of groups (of order one), and hence, by Theorem 2, is a semilattice of completely simple bands. Applying the Suschkewitsch-Rees Theorem to a completely simple band B, we see that the structure group of B must have order one, and hence the law (2) of multiplication reduces to (1).

We come now to the main result of the present paper.

THEOREM 4. *A band of semigroups of type $\mathfrak{T}$ is a semilattice of semigroups each of which is a matrix of semigroups of type $\mathfrak{T}$.*

PROOF. Let S be a band I of semigroups S_α $(\alpha \in I)$ of type $\mathfrak{T}$. By Theorem 3, I is itself a semilattice P of rectangular bands I_τ $(\tau \in P)$. Let ϕ be the natural homomorphism of S onto I, and let ψ be that of I onto P. The mapping $a \to \psi(\phi(a))$ is clearly a homomorphism of S onto P. The inverse image S_τ' of an element τ of P is the class sum of all the S_α for which $\psi(\alpha)=\tau$, i.e. for which $\alpha \in I_\tau$. Describing the elements α of I_τ as pairs (i, κ) of elements of index classes J_τ and K_τ, as above, then we may write $S_{i\kappa}$ for S_α and obtain $S_{i\kappa}S_{j\lambda} \subseteq S_{i\lambda}$ for all $i, j \in J_\tau$ and all $\kappa, \lambda \in K_\tau$. Thus S_τ' is a matrix of semigroups $S_{i\kappa}$, and the latter (being the S_α) are of type $\mathfrak{T}$.

THEOREM 5. *A matrix of* [*completely*] *simple semigroups is* [*completely*] *simple.*

PROOF. Let S be a matrix of simple semigroups $S_{i\kappa}$ $(i \in J, \kappa \in K)$. Let a and b be arbitrary elements of S; we are to show that there exist x, y in S such that $xay=b$. Suppose $a\in S_{i\kappa}$ and $b\in S_{j\lambda}$. Then $bab\in S_{j\lambda}$. Since $S_{j\lambda}$ is simple, there exist u, v in $S_{j\lambda}$ such that $u\cdot bab\cdot v=b$. Hence we may take $x=ub$, $y=bv$.

Now suppose that the semigroups $S_{i\kappa}$ are completely simple. To show that S is completely simple, it suffices to show that any idempotent element e of $S_{i\kappa}$ is primitive in S, as well as in $S_{i\kappa}$. Let f be an idempotent element of S such that $fe=ef=f$. Suppose $f\in S_{j\lambda}$. Then $fe\in S_{j\lambda}S_{i\kappa}\subseteq S_{j\kappa}$ and $ef\in S_{i\kappa}S_{j\lambda}\subseteq S_{i\lambda}$. Hence $S_{j\kappa}$ and $S_{i\lambda}$ must be the same as $S_{j\lambda}$, i.e. $i=j$ and $\kappa=\lambda$. Thus $f\in S_{i\kappa}$, and since e is primitive in $S_{i\kappa}$, we conclude that $f=e$.

From Theorem 5 and the Suschkewitsch-Rees Theorem, it follows that *a semigroup is a matrix of groups if and only if it is completely simple.*

The following is Theorem 9.5 in [1]. (A proof may be constructed for it by suitably modifying the proof of Theorem 2.)[2]

THEOREM (Croisot-Andersen). *The following three assertions concerning a semigroup S are equivalent:*

(A) *S is a class sum of simple semigroups.*

(B) *S is a semilattice of simple semigroups.*

(C) *$a\in Sa^2S$ for every $a\in S$.*

Let us call two elements a, b of a semigroup S r-associate (l-associate) if they generate the same right (left) ideal of S, and $r\cap l$-associate if they are both r- and l-associate. The following is Theorem 7 in [7]:

THEOREM (J. A. Green). *If an element a of a semigroup S is $r\cap l$-associate with a^2, then the set of all elements of S which are $r\cap l$-associate with a is a subgroup of S.*

THEOREM 6. *The following four assertions concerning a semigroup S are equivalent:*

(A) *S is a class sum of completely simple semigroups.*

(B) *S is a class sum of groups.*

(C) *S is a semilattice of completely simple semigroups.*

(D) *$a\in Sa^2\cap a^2S$ for every $a\in S$.*

PROOF. (A) implies (B), since every completely simple semigroup is a class sum of groups. (B) implies (C) by Theorem 2, and evidently (C) implies (A). (B) implies (D), for, if $a\in S$, then a and a^2 both belong to the same subgroup G of S, so that $a\in Ga^2\cap a^2G\subseteq Sa^2\cap a^2S$. (D) implies (B), for, if $a\in S$, (D) implies that a and a^2 are

[2] See footnote 1.

$r \cap l$-associate, and hence a belongs to a subgroup of S by Green's theorem.

THEOREM 7. *A semigroup S is a band of groups if and only if it satisfies the following two conditions*:

(1) $a \in Sa^2 \cap a^2S$ *for every* $a \in S$;

(2) *for every pair of elements* a, b *of* S, $Sba = Sba^2$ *and* $abS = a^2bS$.

PROOF. Let S be a band of groups. In particular, S is a class sum of groups, and (1) follows as in the proof of Theorem 6. Let $a, b \in S$. From the fact that a and a^2 belong to the same subgroup of S, and the assumption that S is a band of groups, we conclude that ba and ba^2 belong to the same subgroup of S, and hence $Sba = Sba^2$. Similarly, $abS = a^2bS$.

Suppose conversely that (1) and (2) hold in S. By (1) and Green's Theorem, S is the class sum of groups G_α $(\alpha \in I)$. Let $a, b \in S$. Let G_α be the subgroup of S to which a belongs, let e_α be the identity element of G_α, and let a^{-1} be the inverse of a in G_α. Replacing b by $a^{-1}b$ in the second part of (2), we conclude $e_\alpha bS = abS$. If a' is any other element of G_α, we conclude similarly that $e_\alpha bS = a'bS$, and hence $abS = a'bS$. Evidently $Sa = Sa'$, and hence $Sab = Sa'b$. Consequently ab and $a'b$ are $r \cap l$-associate. Now, in a semigroup S which is the class sum of groups, two elements of S are $r \cap l$-associate if and only if they belong to the same subgroup of S. Consequently ab and $a'b$ belong to the same subgroup of S. By the left-right dual of this argument, ba and ba' also belong to the same subgroup of S. Thus the relation of belonging to the same subgroup of S is a congruence relation, whence S is a band of groups.

THEOREM 8.[3] *A semigroup S is a semilattice of groups if and only if it satisfies the following two conditions*:

(1) $a \in Sa^2 \cap a^2S$ *for every* $a \in S$;

(2) *if e and f are idempotent elements of S, then* $ef = fe$.

PROOF. Assume that S is a semilattice I of groups G_α $(\alpha \in I)$. Since S is in particular a class sum of groups, (1) follows as in the proof of Theorem 6. To show (2), let e and f be idempotent elements of S. Then $e \in G_\alpha$ and $f \in G_\beta$ for some $\alpha, \beta \in I$. Since G_α and G_β are groups, e and f must be the identity elements thereof: $e = e_\alpha$, $f = e_\beta$. Let $\gamma = \alpha\beta$ $(= \beta\alpha$ since I is by hypothesis commutative). Let e_γ be the identity element of G_γ. Since $e_\alpha e_\beta \in G_\gamma$ we have $e_\gamma \cdot e_\alpha e_\beta = e_\alpha e_\beta$.

Now $e_\gamma e_\alpha \in G_{\gamma\alpha} = G_\gamma$, so that $e_\gamma e_\alpha \cdot e_\gamma = e_\gamma e_\alpha$. Hence

[3] (Added in proof.) This result has also been noted by R. Croisot [8, p. 375].

$$(e_\gamma e_\alpha)^2 = (e_\gamma e_\alpha \cdot e_\gamma)e_\alpha = e_\gamma e_\alpha e_\alpha = e_\gamma e_\alpha.$$

Since e_γ is the only idempotent element of G_γ, we conclude that $e_\gamma e_\alpha = e_\gamma$. Similarly, $e_\gamma e_\beta = e_\gamma$. Hence

$$e_\alpha e_\beta = e_\gamma e_\alpha e_\beta = e_\gamma e_\beta = e_\gamma.$$

Thus $e_\alpha e_\beta = e_{\alpha\beta}$ for all α, β in I, whence $e_\beta e_\alpha = e_{\beta\alpha} = e_{\alpha\beta} = e_\alpha e_\beta$.

Conversely, let S be a semigroup satisfying conditions (1) and (2). By (1) and Green's theorem, S is a class sum of groups. From Theorem 3 of [**5**] and condition (2) it then follows that S is a semilattice of groups; moreover, the theorem cited provides an explicit construction for any such S.

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