CONCERNING INTEGRALS

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1. **Introduction.** R. E. Lane [2] has given the following definition of an integral on the interval $[a, b]$ of the function f with respect to the function g. If D is an ordered subdivision $\{x_i\}_{i=0}^{n+1}$ of the interval $[a, b]$, $\sum_D(f, g)$ denotes the sum

$$\sum_{i=1}^{n+1} \frac{1}{2} [f(x_i) + f(x_{i-1})][g(x_i) - g(x_{i-1})].$$

The statement that *f is g-integrable on* $[a, b]$ means that there exists a number J such that for each positive number ϵ there is an ordered subdivision D of $[a, b]$, such that for every refinement D' of D, $|J - \sum_{D'}(f, g)| < \epsilon$. The number J is the integral on $[a, b]$ of f with respect to g, and is denoted by $\int_a^b f dg$. This integral generalizes the Stieltjes integral and has many of its properties, e.g., is an additive function of intervals and a bilinear function of (f, g); if f is g-integrable on $[a, b]$, then g is f-integrable on $[a, b]$ and $\int_a^b g df = gf|_a^b - \int_a^b f dg$. If $g \in BV[a, b]$ and f has only discontinuities of the first kind in $[a, b]$, then f is g-integrable on $[a, b]$ and, in particular, if g is a simple step-function,[1] then

$$\begin{aligned} \int_a^b f dg = \sum_{a < x \leqq b} & \frac{f(x) + f(x-)}{2} [g(x) - g(x-)] \\ & + \sum_{a \leqq x < b} \frac{f(x+) + f(x)}{2} [g(x+) - g(x)]. \end{aligned} \tag{1.1}$$

Suppose g is a nondecreasing function and $\{f_n\}_{n=1}^{\infty}$ a uniformly bounded sequence of simple step-functions converging to a function f in $[a, b] - S$, where S is a subset of $[a, b]$ of "g-length 0." It is to be expected that the methods of F. Riesz [3] can be applied to the sequence $\{\int_a^b f_n dg\}_{n=1}^{\infty}$, and its limit used to define a Riesz type integral of f with respect to g on $[a, b]$.

In this paper we have reached the desired result, but by methods which are in some way even more elementary than those of Riesz.

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[1] The statement that g is a simple step-function means that g is a function on the set of all numbers and if $[a, b]$ is an interval, then there exist a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ and a sequence $\{k_p\}_{p=1}^{n}$ of numbers, such that $g(x) = k_p$ if $k_{p-1} < x < k_p$.

We depend upon approximation to the nondecreasing function g by step-functions, and essentially reduce the question of measure of S to that of the measure of a finite subset of S.

2. **Outer g-length of a number set.** Throughout this paper, g denotes a nondecreasing function on the set of all numbers.

If S is a number set, then the statement that l_gS *is the outer g-length of* S means that l_gS is the largest number k such that if G is a countable collection of segments covering S, then $k \leqq \sum [g(q-)-g(p+)]$, the sum being taken over every segment (p, q) in G.

We state here, without proof, some elementary properties of outer g-length.

(i) The outer g-length of the interval $[a, b]$ is $g(b+)-g(a-)$.

(ii) The outer g-length of the segment (a, b) is $g(b-)-g(a+)$.

(iii) If Q is an open and bounded number set and ϵ a positive number, then there exists a finite collection G of mutually exclusive intervals, such that[2] G^* is a subset of Q, $l_g[a, b]=g(b)-g(a)$ for each interval $[a, b]$ in G, and $0 \leqq l_gQ - l_gG^* < \epsilon$.

(iv) If S and T are bounded and mutually exclusive number sets and h is a nondecreasing simple step-function, then $l_h(S+T)=l_hS+l_hT$.

(v) If each of S and T is a bounded number set, then $l_g(S+T) \leqq l_gS+l_gT$.

THEOREM A. *If S is a bounded number set, each of ϵ and δ a positive number and $l_gS \geqq \delta$, then there exists a nondecreasing simple step-function h, such that, for, every number x, $|h(x)-g(x)| < \epsilon$ and $l_hS \geqq \delta$.*

PROOF. Suppose $[a, b]$ is an interval containing S. There exists an ordered subdivision $\{x_i\}_{i=0}^{n+1}$ of $[a, b]$ such that if $x \in (x_{i-1}, x_i)$ then $|g(x)-g(x_{i-1}+)| < \epsilon/2$. Suppose $\{y_i\}_{i=1}^{n+1}$ is a sequence of numbers such that $y_i \in (x_{i-1}, x_i)$ and $y_i \in S$ if (x_{i-1}, x_i) contains a number belonging to S. There exists a number a' less than a, such that if $x \in (a', a)$ then $|g(x)-g(a-)| < \epsilon/2$ and a number b' greater than b, such that if $x \in (b, b')$ then $|g(x)-g(b+)| < \epsilon/2$.

There exist a simple step-function h_1 on the set of all numbers less than or equal to a', such that if $x \leqq a'$ then $|h_1(x)-g(x)| < \epsilon/2$, $h_1(a')=g(a')$, and h_1 is nondecreasing, and a simple step-function h_2 on the set of all numbers greater than or equal to b', such that if $x \geqq b'$ then $|h_2(x)-g(x)| < \epsilon/2$, $h_2(b')=g(b')$, and h_2 is nondecreasing (cf. [2]).

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[2] If G is a collection of sets, then G^* denotes the set that is the logical sum of the sets in G.

h denotes the simple step-function defined as follows: if $x \leqq a'$, $h(x) = h_1(x)$; if $x \geqq b'$, $h(x) = h_2(x)$; if $a' < x < a$, $h(x) = g(a'+)$; if $b < x < b'$, $h(x) = g(b'-)$, $h(x_i) = g(x_i)$ if $i = 0, 1, \cdots, n+1$; if $y_i \notin S$ and $x_{i-1} < x < x_i$, $h(x) = g(y_i)$; and if $y_i \in S$, $h(x) = g(x_{i-1}+)$ or $h(x) = g(x_i-)$, according as $x_{i-1} < x < y_i$ or $y_i \leqq x < x_i$, respectively.

h is a nondecreasing simple step-function, and if x is a number, then $|h(x) - g(x)| < \epsilon$.

Suppose $l_h S < \delta$.

If S is a subset of $\{x_i\}_{i=0}^{n+1}$ then, inasmuch as $g(x_i+) - g(x_i-) \leqq h(x_i+) - h(x_i-)$, we see that $l_g S \leqq l_h S < \delta$, contrary to the hypothesis of the theorem. Therefore, there is a subset of S in one of the segments (x_{i-1}, x_i). Suppose M denotes the collection of these segments containing subsets of S, and S_1 is the common part of S and M^*. If one of the numbers x_i belongs to S, denote by S_2 the common part of S and $\{x_i\}_{i=0}^{n+1}$ and suppose $k = l_h S_2$. If $S = S_1$ then $k = 0$.

Suppose $\sum_{(p,q) \in M} [g(q-) - g(p+)] < l_g S - k$. Then there exists a positive number t such that $\sum_{(p,q) \in M} [g(q-) - g(p+)] + k + t < l_g S$. If $S = S_1$, so that $k = 0$, this implies $l_g S < l_g S$, an absurdity. If $S \neq S_1$, there is a finite collection H of segments covering S_2 such that $\sum_{(p,q) \in H} [g(q-) - g(p+)] < k + t$, and therefore

$$l_g S \leqq \sum_{(p,q) \in M+H} [g(q-) - g(p+)] < l_g S,$$

an absurdity. Consequently, we see that

$$k + \sum_{(p,q) \in M} [g(q-) - g(p+)] \geqq l_g S \geqq \delta.$$

If $S = S_1$, this states that $l_h S \geqq \delta$.

If $S \neq S_1$, this states that $l_h S_2 + l_h S_1 \geqq \delta$ so that, by (iv), $l_h S \geqq \delta$.

Thus, the supposition $l_h S < \delta$ is false, and Theorem A is established.

3. **Sequences of simple step-functions.** The following theorem is along the lines of a theorem of Egoroff [1].

THEOREM B. *If S is a proper subset of the interval $[a, b]$, $l_g S = 0$, $\{h_n\}_{n=1}^{\infty}$ is a sequence of step-functions such that, for each number x in $[a, b] - S$, $h_n(x) \to 0$ as $n \to \infty$, and each of ϵ and δ is a positive number, then there exists a subset T of $[a, b] - S$ and a positive integer N such that, for each x in T and each integer n greater than N, $|h_n(x)| < \epsilon$ and $l_g T \geqq l_g [a, b] - \delta$.*

PROOF. Suppose there exists a positive number ϵ and a positive number δ, such that if N is a positive integer, T a subset of $[a, b] - S$ and, for each integer n greater than N and each x in T, $|h_n(x)| < \epsilon$, then $l_g T < l_g [a, b] - \delta$.

If n is a positive integer, then there exists a number x in $[a, b] - S$

such that for some integer m greater than n, $|h_m(x)| \geqq \epsilon$. Otherwise, the set $[a, b]-S$ is a set T for which the above supposition is violated. For each positive integer n, U_n denotes the set such that $x \in U_n$ only if $x \in [a, b]-S$ and $|h_m(x)| \geqq \epsilon$ for some integer m greater than n. We see that U_{n+1} is a subset of U_n. We shall prove that there exists a nonempty and closed number set C_n such that C_n is a subset of U_n and C_{n+1} is a subset of C_n. If $y \in C_n$ for $n=1, 2, 3, \cdots$, then $y \in [a, b]-S$ and $h_n(y)$ not$\rightarrow 0$ as $n \rightarrow \infty$, contrary to the hypothesis of the theorem. This contradiction will show that our supposition is false and the theorem will then be established.

If $r_n = l_g U_n$, then $\{r_n\}_{n=1}^{\infty}$ is a nonincreasing sequence with a nonnegative limit r. If $r=0$ and k is a positive integer such that $r_k < \delta$, the set $[a, b]-[S+U_k]$ is a set T for which our supposition is violated. Consequently, $r>0$.

H denotes the set such that $x \in H$ only if x is a, b, or a number in $[a, b]$ where, for some positive integer n, h_n is not continuous. If H is finite, $K=H$; if H is infinite, K is a finite subset of H containing a and b, such that $\sum_{x \in H-K}[g(x+)-g(x-)] < r\theta$, where θ is a positive number less than $1/4$. If there is a number y such that $y \in K$ and $y \in U_n$ for $n=1, 2, 3, \cdots$, we take $C_n=(y)$ for $n=1, 2, 3, \cdots$.

Suppose there is a positive integer k such that K and U_k have no common part. If n is an integer greater than k, U_n is not a subset of H; U_n' denotes $U_n - H \cdot U_n$; Q_n denotes the set such that $x \in Q_n$ if and only if x is in the segment (a, b) and there is an integer m greater than n such that h_m is continuous at x and $|h_m(x)| \geqq \epsilon$. Q_n is open, $Q_{n+1} \subset Q_n$, and $U_n' = Q_n - (H+S) \cdot Q_n$; if $x \in K$ and $x \notin S$, then $x \notin Q_n$.

There exists a finite collection G_1 of mutually exclusive intervals such that if $[p, q] \in G_1$ then g is continuous at p and at q, $G_1^* \subset Q_{k+1}$, and $0 \leqq l_g Q_{k+1} - l_g G_1^* < r\theta$, so that $l_g G_1^* > r - r\theta$. If i is a positive integer less than $k+2$, C_i' denotes the closed set G_1^*. If i is an integer greater than 1, then $l_g(Q_{k+1} \cdot G_1^*) \geqq r - r\theta$.

For each integer i greater than 1, there exists a finite collection G_i of mutually exclusive intervals such that if $[p, q] \in G_i$, then g is continuous at p and at q, $G_i^* \subset Q_{k+1} \cdot G_{i-1}^*$, $0 \leqq l_g(Q_{k+i} \cdot G_{i-1}^*) - l_g G_i^* < r\theta^i$ and, for each integer j greater than i, $l_g(Q_{k+j} \cdot G_i^*) \geqq r - r\theta - r\theta^2 - \cdots - r\theta^i$. For each positive integer i, C_{k+i}' denotes the closed set G_i^*. Suppose Q is an open set covering $(H+S) \cdot Q_k$ such that $l_g Q < r\theta$. Since $l_g C_{k+i}' > r(1-2\theta)/(1-\theta) > r\theta$, then C_n' is not a subset of Q ($n=1, 2, 3, \cdots$). If C_n is the closed set $C_n' - C_n' \cdot Q$, then $U_n \supset C_n \supset C_{n+1}$, and Theorem B is established.

4. g-**summable functions.** In this section we consider functions on an interval $[a, b]$, and suppose the nondecreasing function g is such

that $g(x)=g(a)$ for $x<a$ and $g(x)=g(b)$ for $x>b$.

THEOREM C. *If S is a proper subset of the interval $[a, b]$, $l_gS=0$, $\{f_n\}_{n=1}^{\infty}$ a sequence of simple step-functions, uniformly bounded on $[a, b]$, which converges to 0 on $[a, b]-S$, $f_n(x+)\to 0$ as $n\to\infty$ if $a\leqq x<b$ and $g(x+)>g(x)$ and $f_n(x-)\to 0$ as $n\to\infty$ if $a<x\leqq b$ and $g(x)>g(x-)$, then*

$$\int_a^b f_n dg \to 0 \qquad \textit{as } n\to\infty.$$

PROOF. Suppose M is a number such that if $x\in[a, b]$ and n is a positive integer, then $|f_n(x)|<M$. Suppose ϵ is a positive number and $\epsilon_1=\epsilon/\{4+8[g(b)-g(a)]\}$ and $\delta_1=\epsilon/2M$.

There exists a subset T_1 of $[a, b]-S$ and a positive integer N_1 such that, if $n>N_1$ and $x\in T_1$, then $|f_n(x)|<\epsilon_1$ and $l_gT_1\geqq l_g[a, b]-(\delta_1/2)$. Suppose H is the set such that $x\in H$ if and only if $x\in[a, b]$ and g is not continuous at x or, for some positive integer n, f_n is not continuous at x. There exists a finite subset K of H such that

$$\sum\nolimits_{x\in H-K}[g(x+)-g(x-)]<\delta_1/2.$$

If $T=T_1-(H-K)\cdot T_1$, then $l_gT+l_g[a, b]-\delta_1$.

There exists a sequence $\{h_n\}_{n=0}^{\infty}$ of nondecreasing simple step-functions such that $h_n(a)=g(a)$, $h_n(b)=g(b)$, $l_{h_n}T\geqq l_g[a, b]-\delta_1$ or $l_{h_n}\{[a, b]-T\}\leqq\delta_1$ and, for each number x in $[a, b]$, $|g(x)-h_n(x)|<1/n$.

Now, if each of m and n is a positive integer (cf. (1.1))

$$\int_a^b f_m dh_n = \sum_{x\in T\cdot(a,b]}\frac{f_m(x)+f_m(x-)}{2}[h_n(x)-h_n(x-)]$$
$$+\sum_{x\in T\cdot[a,b)}\frac{f_m(x+)+f_m(x)}{2}[h_n(x+)-h_n(x)]+\sum_3,$$

where $|\sum_3|\leqq M\delta_1=\epsilon/2$, $\sum_3$ being a sum of like terms taken for $x\in[a, b]-T$. If we consider separately those terms for which $x\in K\cdot T$ and for which $x\in T-K\cdot T$, we see that there exists a positive integer N such that if $m, n>N$, then $|\int_a^b f_m dh_n|<\epsilon$.

Now, $\int_a^b f_m dg=\int_a^b f_m dh_n+\int_a^b f_m d(g-h_n)$ so that, if we use integration by parts, and $m, n>N$,

$$\left|\int_a^b f_m dg\right|\leqq\left|\int_a^b f_m dh_n\right|+\left|\int_a^b (g-h_n)df_m\right|\leqq\epsilon+\frac{1}{n}V_a^b f_m$$

or $\int_a^b f_m dg\to 0$ as $m\to\infty$.

This completes the proof of Theorem C.

The statement that *the function f is g-summable on* $[a, b]$ means that there exists a sequence $\{f_m\}_{m=1}^{\infty}$ of simple step-functions, uniformly bounded on $[a, b]$, such that $f_m(x) \to f(x)$ as $m \to \infty$ for every number x in $[a, b]$ or in $[a, b] - S$, where S is a subset of $[a, b]$ of outer g-length 0, and $f_m(x-) \to f(x-)$ as $m \to \infty$ if $a < x \leqq b$ and $g(x) > g(x-)$, and $f_m(x+) \to f(x+)$ as $m \to \infty$ if $a \leqq x < b$ and $g(x+) > g(x)$.

We see by Theorem C that if f is g-summable on $[a, b]$, then there exists a number J such that if $\{f_m\}_{m=1}^{\infty}$ is any sequence of simple step-functions having the above properties:

$$\int_a^b f_m dg \to J \qquad \text{as } m \to \infty .$$

We define the number J to be the integral $\int_a^b f dg$ on $[a, b]$ of f with respect to g.

It is easy to show that if $\{f_m\}_{m=1}^{\infty}$ is a uniformly bounded sequence of g-summable functions converging in the manner described in the above definition to a function f, then f is g-summable and $\int_a^b f_m dg \to \int_a^b f dg$ as $m \to \infty$.

Remark added in proof. My attention has been called to the fact that the definition I accredited to Lane was given by H. L. Smith, *On the existence of the Stieltjes integral* (Trans. Amer. Math. Soc. vol. 27 (1925) pp. 491–495).

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