NOTE ON A THEOREM OF OSTROWSKI[1]

ALFRED HUBER

Generalizing a result of W. F. Osgood [1], A. M. Ostrowski [2] proved the following theorem: *Let $f(t, x)$ be a continuous function of the point (t, x) for $a<x<b$ and $t \geqq T$ and suppose that we have*

$$\lim_{t \to \infty} f(t, x) = f(x),$$

where $f(x)$ is also continuous in (a, b); then for any $\epsilon>0$ there exists a subinterval J of (a, b) and a number T_0 such that we have for $x \in J$, $t \geqq T_0$:

$$|f(t, x) - f(x)| < \epsilon.$$

It is the purpose of this note to show that even the following statement is true:

THEOREM. *Let the function $f(t, x)$ be defined for $a<x<b$, t varying on a set E of real numbers, which is assumed to be unbounded from above, but otherwise arbitrary; let $f(t, x)$ be continuous in x for fixed t and suppose that we have*

$$\lim_{t \to \infty} f(t, x) = f(x),$$

where $f(x)$ is also continuous in (a, b). Then for any $\epsilon>0$ there exists a subinterval J of (a, b) and number N such that we have for all $x \in J$, $t \in E_N$, where $E_N = E \cap [t \geqq N]$,

$$|f(t, x) - f(x)| < \epsilon.$$

Our proof is indirect. The opposite statement can be formulated as follows:

There exists a number $\epsilon_0>0$ with the following property: Given any subinterval J of (a, b) and an arbitrary number N, there always exists a point $x' \in J$ and a number $t' \in E_N$ such that

$$|f(t', x') - f(x')| > \epsilon_0$$

holds. (Clearly we may require x' to be an interior point of J.)

Let us now consider an arbitrary closed subinterval $\langle a_1, b_1 \rangle$ of (a, b). There exists a point x_1, interior to $\langle a_1, b_1 \rangle$, and a number $t_1 \in E_1$

Received by the editors October 5, 1953.

[1] This research was supported in whole by the United States Air Force under Contract No. AF 18(600)-573 monitored by the Office of Scientific Research.

such that

$$(1) \qquad |f(t_1, x_1) - f(x_1)| > \epsilon_0$$

is fulfilled. Since both $f(x)$ and $f(t_1, x)$ are continuous functions of x, (1) remains valid in a sufficiently small neighborhood U_1 of x_1 if we keep t_1 fixed. Let $\langle a_2, b_2\rangle$ denote a closed interval, contained in both U_1 and $\langle a_1, b_1\rangle$.

There exists a point x_2, interior to $\langle a_2, b_2\rangle$, and a number $t_2 \in E_2$ such that

$$(2) \qquad |f(t_2, x_2) - f(x_2)| > \epsilon_0$$

holds. (2) remains valid in a neighborhood U_2 of x_2. Let $\langle a_3, b_3\rangle$ denote a closed interval, contained in both U_2 and $\langle a_2, b_2\rangle$. And so forth.

We thus obtain a nested sequence of closed intervals. They have—at least—one point x^* in common. We have

$$(3) \qquad |f(t_n, x^*) - f(x^*)| > \epsilon_0 \qquad (n = 1, 2, 3, \cdots).$$

Since $t_n \to \infty$ for $n \to \infty$, (3) is a contradiction to the hypothesis that $\lim_{t\to\infty} f(t, x^*) = f(x^*)$. This proves the theorem.

The proof is still valid, if—as pointed out by J. B. Diaz—the continuity of $f(t, x)$ and $f(x)$ is replaced by the following weaker condition: $|f(t, x) - f(x)|$ is lower semicontinuous in x for any fixed t.

The case of a finite limiting point for t can be treated in an analogous way.

References

1. W. F. Osgood, *Non-uniform convergence and the integration of series term by term*, Amer. J. Math. vol. 19 (1897) p. 161.

2. A. M. Ostrowski, *Generalization of a theorem of Osgood to the case of continuous approximation*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 648–649.

University of Maryland