Finally if λ is such a sequence that

$$\lambda_n = 2\cdot 3^{-1}\cdot 3^{-n} \qquad \text{whenever } n \in \omega$$

then subsum λ is the Cantor ternary set of Lebesgue measure zero.

UNIVERSITY OF CALIFORNIA, BERKELEY

BOUNDS FOR THE MODULI OF THE ZEROS OF A POLYNOMIAL

FULTON KOEHLER

1. Introduction. Let $f(z)=a_0+a_1z+\cdots+a_nz^n$, $a_0\neq 0$, $a_n\neq 0$. Following the notation of Ostrowski [1], we define the Newton diagram of $f(z)$ as the broken line D in the (x, y)-plane with the following properties:

(1) D extends from the point $x=0$, $y=-\log |a_0|$ to the point $x=n$, $y=-\log |a_n|$;

(2) $y_\nu\leqq(y_{\nu-1}+y_{\nu+1})/2$ $(\nu=1, 2, \cdots, n-1)$, where y_ν is the ordinate of D for $x=\nu$;

(3) The points $(\nu, -\log |a_\nu|)$ $(\nu=0, 1, \cdots, n)$ all lie on or above D;

(4) If (ν, y_ν) is a corner point of D, i.e., $y_\nu<(y_{\nu-1}+y_{\nu+1})/2$, then $y_\nu=-\log |a_\nu|$.

Values of ν for which $(\nu, -\log |a_\nu|)$ is a corner point or an end point of D are called principal indices of $f(z)$. Let $y_\nu=-\log T_\nu$ so $T_\nu=|a_\nu|$ when ν is a principal index, and let $R_\nu=T_{\nu-1}/T_\nu$. The slope of D in the interval $(\nu-1, \nu)$ is $\log R_\nu$.

Let the zeros of $f(z)$ be denoted by $z_1, z_2, \cdots, z_n$ with $0<|z_1|\leqq|z_2|\leqq\cdots\leqq|z_n|$, and let b_k^n and B_k^n $(k=1, 2, \cdots, n)$ be the minimum and maximum respectively of $|z_k|/R_k$ for all polynomials of degree n which do not vanish at the origin. Ostrowski [1] has shown, by considering the polynomial $z^nf(1/z)$, that

$$B_k^n = 1/b_{n-k+1}^n. \tag{1}$$

Ostrowski proved that b_1^n is the positive root of the equation $1=x+x^2+\cdots+x^n$, that b_n^n is equal to $1/n$, and that

Presented to the Society, April 24, 1953; received by the editors March 10, 1953 and, in revised form, September 28, 1953.

$$(2) \qquad |z_k|/R_k > 1 - (1/2)^{1/k} \qquad (k = 2, 3, \cdots, n-1).$$

He proved this last inequality for the zeros of a Taylor's series, with $>$ replaced by $\geqq$, and for all values of k, and showed that in this case the bound is the best possible. He also proved that $b_2^3 = 1/3$.

The main result to be proved in this paper is that b_k^n is equal to the positive root of the equation

$$(3) \qquad 1 = \begin{bmatrix} k \\ 1 \end{bmatrix} x + \begin{bmatrix} k \\ 2 \end{bmatrix} x^2 + \cdots + \begin{bmatrix} k \\ n-k+1 \end{bmatrix} x^{n-k+1} = \phi_k^n(x),$$

where

$$\begin{bmatrix} k \\ p \end{bmatrix} = \binom{k+p-1}{p}.$$

It may be noted that ϕ_k^n is a partial sum of the series for $(1-x)^{-k}-1$, and that the equation $(1-x)^{-k}-1=1$ has as its only positive root the value $x=1-(1/2)^{1/k}$ which appears in (2).

A problem which will be seen to be related to this result is the following, considered by Van Vleck [2]:[1] given $a_0, a_1, \cdots, a_{k-1}$, $a_h \neq 0$ $(k \leqq h \leqq n)$, to prove the existence of an upper bound for $|z_k|$. Van Vleck proves the existence of such an upper bound for any h, and he shows that, when $h=k$, such a bound will be the positive root ρ of the equation[2]

$$|a_k| \rho^k = \sum_{\lambda=0}^{k-1} \binom{n-\lambda}{k-\lambda} |a_\lambda| \rho^\lambda.$$

In this paper we give a similar explicit determination of such a bound for any h.

We state here two lemmas which are proved in Ostrowski's paper.

LEMMA 1. *If k and l are two consecutive principal indices $(k<l)$, then*

$$\left|\frac{a_k}{a_l}\right| = R_l^{l-k},$$

$$\left|\frac{a_\nu}{a_k}\right| \leqq R_l^{k-\nu}, \qquad \left|\frac{a_l}{a_\nu}\right| \geqq R_l^{\nu-l} \qquad (k \leqq \nu \leqq l),$$

$$\left|\frac{a_\nu}{a_l}\right| < R_l^{l-\nu}, \qquad \left|\frac{a_\nu}{a_k}\right| < R_l^{k-\nu} \qquad (\nu < k, \nu > l).$$

[1] The author wishes to thank the referee for calling to his attention the paper of Van Vleck.

[2] See also [3, p. 110 and p. 115].

LEMMA 2. *By a transformation of the form* $g(z)=af(bz)$, *which leaves the ratios* $|z_k|/R_k$ *unchanged, the segment of the Newton diagram for* $g(z)$ *in any unit interval* $(\nu-1, \nu)$ *can be made to fall on the x axis.*

2. Let homogeneous polynomials $S_\nu(z_1, z_2, \cdots, z_m)$ and $H_\nu(z_1, z_2, \cdots, z_m)$ be defined by the formulas:

$$(4)\qquad \begin{aligned} (1-z_1t)(1-z_2t)\cdots(1-z_mt) &= \sum_{\nu=0}^{\infty}(-1)^\nu S_\nu t^\nu, \\ (1-z_1t)^{-1}(1-z_2t)^{-1}\cdots(1-z_mt)^{-1} &= \sum_{\nu=0}^{\infty} H_\nu t^\nu. \end{aligned}$$

Multiplication of corresponding members in these two formulas gives

$$(5)\qquad H_\nu - S_1H_{\nu-1} + S_2H_{\nu-2} - \cdots \pm S_\nu = 0 \qquad (\nu = 1, 2, 3, \cdots).$$

Let us consider the polynomial

$$K_{\nu\alpha} = H_\nu - S_1H_{\nu-1} + \cdots + (-1)^\alpha S_\alpha H_{\nu-\alpha} \quad (\nu, \alpha = 0, 1, 2, \cdots).$$

This is a homogeneous polynomial in $z_1, z_2, \cdots, z_m$ of degree ν and is therefore the sum of terms of the form $z_1^{\beta_1}z_2^{\beta_2}\cdots z_m^{\beta_m}$, where $\beta_1, \beta_2, \cdots, \beta_m$ are non-negative integers whose sum is ν. Let λ be the number of the β's which are positive. Then the coefficient of $z_1^{\beta_1}z_2^{\beta_2}\cdots z_m^{\beta_m}$ in $K_{\nu\alpha}$ is equal to

$$1-\binom{\lambda}{1}+\binom{\lambda}{2}-\cdots+(-1)^\alpha\binom{\lambda}{\alpha}$$

where it is understood that

$$\binom{\lambda}{p} = 0 \qquad\qquad \text{if } p > \lambda.$$

It therefore follows that, if similar terms in $K_{\nu\alpha}$ are combined, all coefficients are positive if α is even and negative if α is odd. Since

$$K_{\nu\alpha} = K_{\nu,\alpha-1} + (-1)^\alpha S_\alpha H_{\nu-\alpha},$$

the absolute value of the coefficient of any term in $K_{\nu\alpha}$ is not greater than the coefficient of that term in $S_\alpha H_{\nu-\alpha}$, i.e., $K_{\nu\alpha}$ is dominated by $S_\alpha H_{\nu-\alpha}$. Hence,

$$(6)\qquad |K_{\nu\alpha}| = \left|\sum_{\sigma=0}^{\alpha}(-1)^\sigma S_\sigma H_{\nu-\sigma}\right| \leqq \binom{m}{\alpha}\left[\begin{matrix} m \\ \nu-\alpha \end{matrix}\right] M^\nu$$
$$(\nu, \alpha = 0, 1, 2, \cdots),$$

where

$$M = \max \{ |z_1|, |z_2|, \cdots, |z_m| \}.$$

3. Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ have zeros $z_1, z_2, \cdots, z_n$ with $|z_1| \leqq |z_2| \leqq \cdots \leqq |z_n|$, let $\rho_\nu = 1/z_\nu$, and let $f(z)$ be written in the form

$$\begin{aligned} f(z) &= \sum_{\nu=0}^{n} a_\nu z^\nu \\ &= (\gamma_0 + \gamma_1 z + \cdots + \gamma_{k-1} z^{k-1})(1 - \rho_k z)(1 - \rho_{k+1} z) \cdots (1 - \rho_n z). \end{aligned} \tag{7}$$

By equating coefficients in the equations

$$\begin{aligned} &a_0 + a_1 z + \cdots + a_n z^n \\ &\qquad = (\gamma_0 + \gamma_1 z + \cdots + \gamma_{k-1} z^{k-1})(1 - S_1 z + S_2 z^2 - \cdots), \\ &\gamma_0 + \gamma_1 z + \cdots + \gamma_{k-1} z^{k-1} \\ &\qquad = (a_0 + a_1 z + \cdots + a_n z^n)(1 + H_1 z + H_2 z^2 + \cdots), \end{aligned}$$

where $S_i = S_i(\rho_k, \rho_{k+1}, \cdots, \rho_n)$ and $H_i = H_i(\rho_k, \rho_{k+1}, \cdots, \rho_n)$ are defined as in (4), we obtain

$$a_\nu = \sum_{\sigma=0}^{\nu} (-1)^\sigma S_\sigma \gamma_{\nu-\sigma} \qquad (\nu = 0, 1, 2, \cdots, n) \tag{8}$$

and

$$\gamma_\nu = \sum_{\sigma=0}^{\nu} H_\sigma a_{\nu-\sigma} \qquad (\nu = 0, 1, 2, \cdots, n), \tag{9}$$

letting $\gamma_\nu = 0$ for $\nu \geqq k$.

For $\nu \geqq k$ we can obtain from (8) and (9) an expression for a_ν in terms of $a_0, a_1, \cdots, a_{k-1}$ as follows:

$$\begin{aligned} a_\nu &= \sum_{\sigma=\nu-k+1}^{\nu} (-1)^\sigma S_\sigma \gamma_{\nu-\sigma} = \sum_{\sigma=\nu-k+1}^{\nu} \sum_{\rho=0}^{\nu-\sigma} (-1)^\sigma S_\sigma H_\rho a_{\nu-\sigma-\rho} \\ &= \sum_{\lambda=0}^{k-1} \sum_{\sigma=\nu-k+1}^{\nu-\lambda} (-1)^\sigma S_\sigma H_{\nu-\lambda-\sigma} a_\lambda \\ &= - \sum_{\lambda=0}^{k-1} \sum_{\sigma=0}^{\nu-k} (-1)^\sigma S_\sigma H_{\nu-\lambda-\sigma} a_\lambda, \end{aligned} \tag{10}$$

the last step being justified by (5). Hence, by (6),

$$|a_\nu| \leqq \sum_{\lambda=0}^{k-1} \binom{n-k+1}{\nu-k} \begin{bmatrix} n-k+1 \\ k-\lambda \end{bmatrix} \rho^{\nu-\lambda} |a_\lambda|, \tag{11}$$

where $\rho = |\rho_k|$.

The above equation (10) and inequality (11) are given by Van Vleck [2, p. 115, (16)] for the case $\nu = k$. He also gives an equation [2, p. 115, (17)] which is an implicit form of our equation (10). By the use of (11) we can now give an explicit solution for the general problem considered by Van Vleck which was mentioned earlier. Namely, if $a_0, a_1, \cdots, a_{k-1}, a_\nu \neq 0$ $(k \leqq \nu \leqq n)$ are given, then $|z_k|^{-1}$ is at least equal to the positive root ρ of the equation

$$|a_\nu| = \binom{n-k+1}{\nu-k} \sum_{\lambda=0}^{k-1} \begin{bmatrix} n-k+1 \\ k-\lambda \end{bmatrix} |a_\lambda| \rho^{\nu-\lambda}.$$

We now consider the problem of determining b_k^n and B_k^n. Let n be considered fixed and let $\phi_k(x) = \phi_k^n(x)$ be defined by (3) and let r_k be the positive root of the equation $\phi_k(x) = 1$. Since $\phi_k(x)$ is monotonically increasing for $x > 0$, the conditions $x > 0$, $\phi_k(x) \geqq 1$ imply that $x \geqq r_k$.

By Lemma 2, we may assume without loss of generality that the Newton diagram for $f(z)$ lies along the x-axis in the interval $(k-1, k)$. Then $R_k = 1$, $|a_\nu| = 1$ for some $\nu \geqq k$ and, by Lemma 1, $|a_h| \leqq 1$ for all h. With this value of ν, (11) gives

$$\begin{aligned} 1 &\leqq \sum_{\lambda=0}^{k-1} \binom{n-k+1}{\nu-k} \rho^{\nu-k} \begin{bmatrix} n-k+1 \\ k-\lambda \end{bmatrix} \rho^{k-\lambda} \\ &= \binom{n-k+1}{\nu-k} \rho^{\nu-k} \phi_{n-k+1}(\rho). \end{aligned} \tag{12}$$

It is clear from the definition that $r_k \leqq 1/k$ since the first term in $\phi_k(x)$ is kx. If $\rho < r_{n-k+1} \leqq (n-k+1)^{-1}$, then

$$\binom{n-k+1}{\nu-k} \rho^{\nu-k} \leqq 1,$$

$\phi_{n-k+1}(\rho) < 1$, and there is a contradiction of (12). Hence $\rho \geqq r_{n-k+1}$, $|z_k| \leqq r_{n-k+1}^{-1}$.

In order to show that $|z_k|/R_k$ can actually equal r_{n-k+1}^{-1}, we define $f(z)$ by (7) with $\rho_k = \rho_{k+1} = \cdots = \rho_n = r_{n-k+1}$, $a_0 = a_1 = \cdots = a_{k-1} = 1$. We then have, from (9),

$$\begin{aligned} a_k &= -\sum_{\sigma=1}^{k} H_\sigma a_{k-\sigma} = -\sum_{\sigma=1}^{k} \begin{bmatrix} n-k+1 \\ \sigma \end{bmatrix} r_{n-k+1}^{\sigma} \\ &= -\phi_{n-k+1}(r_{n-k+1}) = -1, \end{aligned}$$

and, from (11), for $\nu > k$,

$$| a_\nu | \leqq \binom{n-k+1}{\nu-k} r_{n-k+1}^{\nu-k} \phi_{n-k+1}(r_{n-k+1}) < 1.$$

Hence $R_k = 1$. Since $f(z)$ has $n-k+1$ zeros equal to r_{n-k+1}^{-1}, it can have at most $k-1$ zeros less than r_{n-k+1}^{-1}. Therefore, $|z_k| \geqq r_{n-k+1}^{-1}$ and, from what has been proved before, the equality sign must hold. It follows that

$$B_k^n = r_{n-k+1}^{-1} \tag{13}$$

and, from (1), that

$$b_k^n = r_k \qquad (k = 1, 2, \cdots, n). \tag{14}$$

References

1. Alexandre Ostrowski, *Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent*, Acta Math. vol. 72 (1940) pp. 99–257.

2. Edward B. Van Vleck, *On limits to the absolute values of the roots of a polynomial*, Bull. Soc. Math. France vol. 53 (1925) pp. 105–125.

3. Morris Marden, *The geometry of the zeros of a polynomial in a complex variable*, Mathematical Surveys, vol. 3, New York, American Mathematical Society, 1949.

University of Minnesota