A NOTE ON MONOTONE DEFORMATION-FREE MAPPINGS

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If M is a closed subset of a space S and $h: MxI \rightarrow S$ is a deformation, then h is called a deformation-free mapping of M into S if, for each $0 \leqq t \leqq 1$, $h(Mxt) \cap M = 0$. In the case M is a locally simply connected continuum separating the n-sphere S^n, A is a component of $S^n - M$, and M is deformation free into $\bar{A}$ with a monotone open deformation-free map, then A is uniformly locally simply connected [1, Theorem 3.1]. The purpose of this note is to show that the condition that h be open can be omitted.

The first lemma is a standard result in the theory of covering spaces. The second lemma is a simple consequence of the first (also noted by G. D. Mostow [2] in the case $\pi_1(X) = 0$).

LEMMA 1. *Let X and Y be arcwise connected and locally arcwise connected topological spaces. Let $\tilde{Y}$ be a covering of Y with projection $p: \tilde{Y} \rightarrow Y$ and let $f: X \rightarrow Y$ be onto. If $p_*(\pi_1(Y)) \supset f_*(\pi_1(X))$, then there exists a map $g: X \rightarrow \tilde{Y}$ such that $pg = f$. (p_* and f_* denote the maps induced on the fundamental groups by p and f respectively.)*

LEMMA 2. *If X and Y are as in Lemma 1 and*

(1) Y is locally simply connected,

(2) $f: X \rightarrow Y$ is monotone and onto,

then $f_*: \pi_1(X) \rightarrow \pi_1(Y)$ is onto.

PROOF. Take a covering $\tilde{Y}$ of Y such that $p_*(\pi_1(\tilde{Y})) = f_*(\pi_1(X))$ and factor f as in Lemma 1. Define a map $\beta: Y \rightarrow \tilde{Y}$ as follows: If $y \in Y$, then $f^{-1}(y)$ is a connected set (f is monotone) so that $g(f^{-1}(y))$ is also connected. Since $g(f^{-1}(y))$ is contained in the discrete set $p^{-1}(y)$, it is a single point, and the definition $\beta(y) = g(f^{-1}(y))$ gives a single-valued function. It is almost immediate that β is one-to-one, continuous and open. Since $\tilde{Y}$ is a covering of Y, this implies that $\tilde{Y} = Y$, p is the identity, and $\pi_1(Y) = \pi_1(\tilde{Y}) = f_* \pi_1(X)$, so that f_* is onto.

THEOREM. *If M is a locally simply-connected continuum which separates S^n and which is deformation free into $\bar{A}$ (A a component of $S^n - M$) with a monotone deformation-free mapping, then A is uniformly locally simply connected.*

PROOF. Let $h: MxI \rightarrow \bar{A}$ be a monotone deformation-free mapping.

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Since $\overline{A}$ is compact, showing A to be uniformly simply connected is equivalent to showing A to be locally simply connected relative to $\overline{A}$ [**3**]. Since A is locally simply connected, one must simply show that given $\epsilon>0$ and $x\in M$, there exists a $\delta>0$ such that any continuous 1-sphere in $S(x, \delta)\cap A$ is nullhomotopic in $S(x, \epsilon)\cap A$. ($S(x, \alpha)$ means the open ball of radius α with center $(x, 0)$.)

Let $R(x, \zeta)$ denote the set of points in $MxI-Mx0$ which are a distance less than ζ from x, using a metric in SxI. Since h is a deformation-free map and M is locally simply-connected, it is easy to show that one can choose $\delta'>0$ so that any continuous 1-sphere in $R(x, \delta')$ is homotopic to a point in $h^{-1}(S(x, \epsilon)\cap A)$. Choose $\delta>0$ so that $R(x, \delta')$ contains $h^{-1}(S(x, \delta)\cap A)$.

Let $f(S^1)$ be a continuous 1-sphere in $S(x, \delta)\cap A$. Let T be a connected open set in $S(x, \delta)\cap A$ such that $f(S^1)\subset T$. Since h is monotone $h^{-1}(T)$ is connected, and $h^{-1}(T)\subset R(x, \delta')$ by the choice of δ. Letting $h^{-1}(T)=X$, $T=Y$ and $h|h^{-1}(T)=f$, the conditions of Lemma 2 are satisfied. ($h^{-1}(T)$ is locally arcwise connected because M is, and M is by Theorem 1.1 in [**1**]). Hence there is a map f' which is homotopic (in T) with f and a map $g:S^1\rightarrow h^{-1}(T)$ such that $f'=hg$. Now $g(S^1)$ is nullhomotopic in $h^{-1}(S(x, \epsilon)\cap A)$ so that $f(S^1)$ is nullhomotopic in $S(x, \epsilon)\cap A$ and the theorem is proved.

Bibliography

1. M. L. Curtis, *Deformation-free continua*, Ann. of Math. vol. 57 (1953) pp. 231–247.

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3. S. Eilenberg and R. L. Wilder, *Uniform local connectedness and contractibility*, Amer. J. Math. vol. 44 (1942) pp. 408–426.

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