

k -FOLD IRREDUCIBLE DECOMPOSITION OF A SPACE RELATIVE TO A MAPPING¹

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1. Introduction. Let f be a mapping of a space A onto a space B . Let $N(y, f, A, B)$, $y \in B$, be the number of points in $f^{-1}(y)$. Let k be a positive integer. According to G. T. Whyburn [2],² f is said to be k -fold irreducible provided that for $y \in B$, $N(y, f, A, B) \geq k$, and if F is a closed proper subset of A , then for some $z \in B$, $N(z, f|F, F, B) < k$. It should be noted that if f is 1-fold irreducible, then f is *strongly irreducible* in accordance with the terminology of [3] or *irreducible* in accordance with [2].

In this paper there is introduced the notion of a k -fold irreducible decomposition of a space relative to a mapping as follows.

DEFINITION. Let f be a mapping defined on a compact space A onto a space B . A is said to possess a k -fold irreducible decomposition relative to f provided that there exists a decomposition $A = \sum_{i=1}^k A_i$ where each A_i is a nonempty closed subset of A and the decomposition satisfies the following conditions:

- (i) A_i^0 , the interior (rel. to A) of A_i , is dense in A_i .
- (ii) $A_i^0 \cdot A_j^0 = 0$, $i \neq j$.
- (iii) $f(A_i) = B$ and $f|A_i$ is an irreducible mapping.

We further define $D(k, f, A, B)$ to be the set of all $x \in A$ for which $N(f(x), f, A, B) = k$. When there is no chance of confusion we shall write $D(k, f, A, B)$ as $D(k, f, A)$.

In 1.1 through 1.5 some definitions and results of G. T. Whyburn which will be needed are listed. In the remaining part of §1, we obtain results concerning k -fold irreducible decomposition of a space relative to certain types of mappings. One of the principal results (1.13) states that if f is a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum, then the set of points D at which f is exactly k -to-one is dense in A if and only if A has a k -fold irreducible decomposition relative to f . A corollary of this theorem is that for the same hypothesis as the theorem, if for each $x \in A$, $f^{-1}f(x)$ consists of at least k points, then f is k -fold irreducible if and only if A has a k -fold irreducible decomposition relative to f .

Received by the editors March 26, 1953 and, in revised form, September 8, 1953.

¹ The material for this article was taken from Chapter V of the author's dissertation, *A study of multiplicity functions*, University of Virginia, August 1952.

² Numbers in brackets refer to the bibliography at the end of the paper.

In §2, some examples of k -fold irreducible decomposition are given. It is shown that if f is a quasi-monotone mapping defined on a boundary curve A such that for each $x \in A$, $f^{-1}f(x)$ consists of at least k points, then f is k -fold irreducible if and only if f is k -to-one. Also, if f is a quasi-monotone mapping defined on a boundary curve which has a k -fold irreducible decomposition relative to f , then f is a light open mapping. Finally, it is shown that for an open-mapping f defined on a compact 2-manifold A , A has a k -fold irreducible decomposition relative to f if and only if the degree of f , as defined in [3], is k .

For a general reference to topological terms and results used in this paper see [3]. All spaces are assumed to be metric.

1.1. For any set A in a metric space and any positive k , define:

$e_k(A) = \text{g.l.b.} [\max \delta(A_i)]$ for all decompositions $A = \sum_{i=1}^k A_i$ of A into k nonempty subsets A_i . Here δ stands for diameter.

For f , a mapping defined on a metric space X , define

$$e_k(x) = e_k(f^{-1}f(x)), \quad x \in X.$$

1.2. *If the mapping f defined on X generates an upper-semi-continuous decomposition of X , the function $e_k(x)$ is upper-semi-continuous.*

Hence the theorem applies if X is compact.

1.3. $e_k(x) = 0$ if and only if $N(f(x), f, A) \leq k$.

1.4. *Let f be a mapping defined on a compact space A . Suppose for each $x \in A$, $N(f(x), f, A) \geq k$. Then f is k -fold irreducible if and only if the set of points $x \in A$ such that $N(f(x), f, A) = k$ is dense in A (i.e. $D(k, f, A)$).*

COROLLARY 1. *f is irreducible if and only if the set D of all points $x \in A$ with $x = f^{-1}f(x)$ is dense in A .*

COROLLARY 2. *If f is open, then f is k -fold irreducible if and only if f is k -to-one.*

COROLLARY 3. *If f is open, then f is irreducible if and only if f is a homeomorphism.*

1.5. *Let f be a mapping defined on a compact space A onto a space B . Then there exists a compact set $A' \subset A$ such that $f(A') = B$ and $f|_{A'}$ is an irreducible mapping.*

1.6. **THEOREM.** *Let f be a mapping defined on a compact space A onto a space B . If f possesses a k -fold irreducible decomposition relative to f , then $D(k, f, A, B)$ is dense in A .*

PROOF. We first prove the following.

(*) Let U be an open set in A and let $e > 0$. Then there exists an open set Q in U such that (i) if $q \in Q$, then $f^{-1}f(q)$ contains at least k points and (ii) for each $q \in Q$, $e_k(q) < e$, where $e_k(q)$ is as defined in 1.1.

Toward the end of proving this statement, let $A = \sum_{i=1}^k A_i$ be a k -fold irreducible decomposition relative to f and we suppose further that this decomposition is labeled so that $A_1 \cdot U \neq 0$. Since A_1^0 is dense in A_1 , $A_1^0 \cdot U \neq 0$. Let V_1 be a nonempty open subset of $U \cdot A_1^0$ such that $0 < \delta(V_1) < e$. Let $a_1 \in V_1 \cdot D(1, f, A_1, B)$. It is easy to prove that there exists an open set W_1 in B which contains $f(a_1)$ and such that $f^{-1}(W_1) \cdot A_1 \subset V_1$. Now $f^{-1}(W_1) \cdot A_2^0$ is an open nonempty subset of A_2 . Let V_2 be a nonempty subset of $f^{-1}(W_1) \cdot A_2^0$ such that $\delta(V_2) < e$. Let $a_2 \in V_2 \cdot D(1, f, A_2, B)$. Proceeding as before, there exists an open set W_2 in W_1 such that $W_2 \supset f(a_2)$ and $f^{-1}(W_2) \cdot A_2 \subset V_2$. Note that at this stage $f^{-1}(W_2) \cdot A_i$, $i = 1, 2$, is such that $f[f^{-1}(W_2) \cdot A_i] = W_2$ and $\delta(f^{-1}(W_2) \cdot A_i) < e$.

It is easy to see then by induction that we are able to obtain a set W_k in B such that for $Q_i = f^{-1}(W_k) \cdot A_i$, $i = 1, 2, 3, \dots, k$, it is true that $f(Q_i) = W_k$ and $\delta(Q_i) < e$. It is now easy to verify that if we set $Q = Q_1$, then Q satisfies (i) and (ii) of our preliminary statement (*).

Let D_e be the set of all points $x \in A$ such that $e_k(x) < e$. By (ii) of *, D_e is dense in A for each $e > 0$. Since e_k is upper semi-continuous it follows that the set D of all points x in A such that $e_k(x) = 0$ is also dense in A . By 1.3 for each $x \in D$, $f^{-1}(x)$ contains at most k points. Next let U be an open set in A . There exists an open set Q in U such that Q satisfies (i) and (ii) of *. Since D as defined above is dense in A , there exists an $x \in Q \cdot D$. From the properties of Q and D it follows that $f^{-1}f(x)$ consists of exactly k points. Hence $x \in D(k, f, A, B)$ and $D(k, f, A, B)$ is dense in A .

The following is easy to verify.

1.7. Let f be an open mapping of a space A onto a space B and suppose K is a compact subset of A such that $f(K) = B$ and $f|K$ is an irreducible mapping. Let K^0 be the interior (relative to A) of K . Then if $x \in K^0$, $f^{-1}f(x) \cdot K$ consists of exactly one point.

1.8. Let f be an open mapping defined on a compact space A onto a space B such that $D(k, f, A)$ is dense in A . Then,

(a) $d(f) = k$ where $d(f) = \sup [N(f(x), f, A)]$ for $x \in A$.

(b) $D(k, f, A)$ is open in A .

(c) f is a local homeomorphism at each x in $D(k, f, A)$.

(d) If F is a closed subset of A such that $f(F) = B$ and $f|F$ is irreducible, then F^0 , the interior of F (rel. to A), is dense in F .

(e) Suppose F is a closed subset of A such that $f(F) = B$ and h is a

positive integer such that $1 < h \leq k$, $D(h, f|F, F)$ is dense in F , and F^0 is dense in F . Then there exists a decomposition $F = X + Y$ such that X and Y are closed: X^0 is dense in X ; Y^0 is dense in Y ; $X^0 \cdot Y^0 = 0$; $f(X) = f(Y) = B$; $f|X$ is irreducible; $D(h-1, f|Y, Y)$ is dense in Y .

The proofs of (a), (b), and (c) follow easily from the openness of f .

PROOF. (d). Let us suppose that F^0 is not dense in F . Then there exists an open set $U \subset A$ such that $U \cdot F \neq 0$ and $U \cdot F^0 = 0$. Then either $U \cdot F \cdot D(k, f, A)$ is empty or not and we show that either case leads to a contradiction.

Suppose $U \cdot F \cdot D(k, f, A) \neq 0$. Then, by (b), $W = U \cdot D(k, f, A)$ is a nonempty open set which intersects F and since $D(1, f|F, F)$ is dense in F , there exists a point $y_1 \in W \cdot F \cdot D(1, f|F, F)$. Let $y_1 + y_2 + y_3 + \dots + y_k = f^{-1}f(y_1)$ where we note that $y_2 + y_3 + \dots + y_k \subset A - F$. By (c), f is a local homeomorphism at each y_i . Hence, there exist open sets U_i , $i = 1, 2, \dots, k$, such that: $U_i \cdot U_j = 0$ for $i \neq j$, $f(U_i) = f(U_j)$, $f^{-1}f(U_i) = \sum_{i=1}^k U_i$, $f|U_i$ is a homeomorphism, $U_1 \subset U$, $U_i \subset A - F$ for $i = 2, 3, \dots, k$. Since $U \cdot F^0 = 0$, there exists a $q \in U_1 \cdot (A - F)$, whence $f^{-1}f(q) \cdot F = 0$ and we have a contradiction since $f(F) = B$. Next, consider the case in which $U \cdot F \cdot D(k, f, A, B) = 0$. There exists a $y \in U \cdot F \cdot D(1, f|F, F)$. Since $D(k, f, A)$ is dense in A , there exists a sequence $y_i \rightarrow y$ such that $y_i \in U \cdot D(k, f, A)$. By the hypothesis for this case, none of the y_i 's are in F . However, since $f(F) = B$ and since F is compact, there exists a sequence $x_{n_i} \rightarrow z \in F$ such that $x_{n_i} \in F$ and $f(x_{n_i}) = f(y_{n_i})$. Then since $z \in F$ and $y \in F \cdot D(1, f|F, F)$, it follows that $z = y$. Notice that if $y_{n_j} \in D(k, f, A, B)$, each point of $f^{-1}f(y_{n_j})$ is also. Hence for some j , $x_{n_j} \in U \cdot F \cdot D(k, f, A, B)$, a contradiction to the hypothesis for this case.

PROOF OF (e). There exists a closed subset $X \subset F$ such that $f(X) = B$ and such that $f|X$ is an irreducible mapping. Let Y be the closure of $F - X$. We first show that $f(Y) = B$. Suppose $f(Y) \neq B$. Then $f^{-1}(B - f(Y))$ is a nonempty open set such that $f^{-1}(B - f(Y)) \cdot Y = 0$. This leads to a contradiction, for by (d), X^0 the interior (rel. to A) of X is dense in X and further by hypothesis $D(h, f|F, F)$ is dense in F . Hence there exists an $x \in f^{-1}(B - f(Y)) \cdot X^0 \cdot D(h, f|F, F)$. By 1.7, $f^{-1}f(x) \cdot X = x$. Hence $f^{-1}f(x) \cdot (F - X)$ must contain $h-1$ points. Thus, $f^{-1}f(x) \cdot Y \neq 0$ and we have a contradiction.

We proceed to prove that $D(h-1, f|Y, Y)$ is dense in Y . First note that Y^0 , the interior (rel. to A) of Y is dense in Y . Hence, we need show only that $D(h-1, f|Y, Y)$ is dense in Y^0 . Let U be an open set in Y^0 . Since X^0 is dense in X and $f(X) = B$, then $f(X^0)$ is dense in B . Then, since f is open, it is easy to see that $f^{-1}f(X^0)$ is dense in A . Hence $U \cdot f^{-1}f(X^0)$ is a nonempty open subset and since

$D(h, f|F, F)$ is dense in F , it follows that there exists $x \in U \cdot f^{-1}f(X^0) \cdot D(h, f|F, F)$. By 1.7, $f^{-1}f(x) \cdot X$ is a single point, whence since $x \in D(h, f|F, F)$, $x \in D(h-1, f|Y, Y)$.

1.9. Let f be an open mapping defined on a compact space A onto a space B . Let k be a positive integer. Then if $D(k, f, A)$ is dense in A , there exists a k -fold irreducible decomposition of A relative to f .

PROOF. If $k=1$, the theorem is trivial. Let $k>1$. By 1.8 (e), there exists a decomposition $A = A_1 + A_2$ such that A_1, A_2 are closed subsets of A , $A_1^0 \cdot A_2^0 = 0$, $f(A_1) = f(A_2) = B$, $f|A_1$ is irreducible, and $D(k-1, f|A_2, A_2)$ is dense in A_2 . Let L be the collection of all integers m between 2 and k inclusive satisfying the condition that there exists a decomposition $A = \sum_{i=1}^m X_i$ with the following properties: (1) each X_i is closed; (2) $X_i^0 \cdot X_j^0 = 0$ for $i \neq j$; (3) $f(X_i) = B$ and $f|X_i$ is irreducible for $i=1, 2, 3, \dots, m-1$; (4) $f(X_m) = B$ and $f|X_m$ satisfies the condition that $D(k-m+1, f|X_m)$ is dense in X_m ; (5) X_i^0 is dense in X_i for $i=1, 2, \dots, m$. Since $A = A_1 + A_2$ satisfies the above conditions, $2 \in L$. Let $j = \max L$. Now $j = k$, for suppose $j < k$. Then there exists a decomposition $A = Y_1 + Y_2 + \dots + Y_j$ satisfying conditions (1) through (5). But $f(Y_j) = B$ and $D(k-j+1, f|Y_j)$ is dense in Y_j where it is to be noted that $k-j+1 \geq 2$. Hence 1.8 (e) applies to $f|Y_j$ and there exists a decomposition $Y_j = Y_j^* + Y_{j+1}^*$ such that the following conditions are satisfied: $f(Y_j^*) = f(Y_{j+1}^*) = B$; $f|Y_j^*$ is irreducible; $f|Y_{j+1}^*$ is such that $D(k-j, f|Y_{j+1}^*)$ is dense in Y_{j+1}^* ; $Y_j^* \cdot Y_{j+1}^* = 0$; Y_{j+1}^{*0} is dense in Y_{j+1}^* . But then $A = Y_1 + Y_2 + \dots + Y_j^* + Y_{j+1}^*$ satisfies the 5 conditions and $\max L \geq j+1$, a contradiction.

The following remark is easy to verify.

1.10. Let f be a mapping defined on a compact space A and suppose f_1 and f_2 are any continuous factors of f . Then f is irreducible if and only if f_1 and f_2 are irreducible mappings.

1.11. Let f be a mapping defined on a compact space A onto a space B and let f_1 and f_2 be monotone-light factors of f . If $D(k, f, A, B)$ is a dense subset of A , then $D(1, f_1, A, f_1(A))$ is a dense subset of A and $D(k, f_2, f_1(A), B)$ is a dense subset of $f_1(A)$.

PROOF. Because of the properties of the monotone-light factors of a mapping, $D(k, f, A, B) \subset D(1, f_1, A, f_1(A))$. Hence $D(1, f_1, A, f_1(A))$ is dense in A . Also since $D(k, f, A, B)$ is dense in A , $f_1(D(k, f, A, B))$ is dense in $f_1(A)$. But for each $z \in f_1(D(k, f, A, B))$, $N(f_2(z), f_2, f_1(A), B) = k$. Hence $f_1(D(k, f, A, B)) \subset D(k, f_2, f_1(A), B)$ and thus $D(k, f_2, f_1(A), B)$ is also dense in $f_1(A)$.

1.12. Let f be a mapping on a compact space A onto a space B and suppose f_1, f_2 are monotone-light open factors of f . Then if $D(k, f, A, B)$ is dense in A , A has a k -fold irreducible decomposition relative to f .

PROOF. Let $f_1(A) = X$. By 1.11, $D(1, f_1, A, X)$ is dense in A and $D(k, f_2, X, B)$ is dense in X . Hence f_1 is irreducible and, by 1.9, X has a k -fold irreducible decomposition relative to f_2 . Let $X = \sum_{i=1}^k X_i$ be a k -fold decomposition of X relative to f_2 . Let A_i be the closure of $f_1^{-1}(X_i^0)$ for $i=1, 2, \dots, k$. We show that $\sum_{i=1}^k A_i$ is a k -fold decomposition of A relative to f . Since $f_1(\sum_{i=1}^k A_i) = X$ and f_1 is irreducible, it follows that $\sum_{i=1}^k A_i = A$. From the definition of A_i , $f_1^{-1}(X_i^0)$ is dense in A_i . Then since $A_i^0 \supset f_1^{-1}(X_i^0)$, A_i^0 is also dense in A_i . Further, it is easy to see that $A_i^0 \cdot A_j^0 = 0$ for $i \neq j$. Finally, we show that $f(A_i) = B$ and $f|A_i$ is an irreducible mapping. $f_1(A_i) \supset f_1 f_1^{-1}(X_i^0) = X_i^0$. Since $f_1(A_i)$ is closed and X_i^0 is dense in X_i , $f_1(A_i) = X_i$. So $f_2 f_1(A_i) = f_2(X_i) = B$. Also, $A_i^0 \cdot D(1, f_1, A, X) \subset D(1, f_1|A_i, A_i, X_i)$. Then since $A_i^0 \cdot D(1, f_1, A, X)$ is dense in A_i , so also is $D(1, f_1|A_i, A_i, X_i)$. Thus $f_1|A_i$ is irreducible and since $f_2|X_i$ is also, it follows that $f|A_i$ is as well.

As a corollary to the above proof we have the following

COROLLARY. *If f is a mapping defined on a compact space A with monotone light open factorization $f_2 f_1$ and if f_1 is irreducible and the space $f_1(A)$ possesses a k -fold irreducible decomposition relative to f_2 , then A possesses a k -fold irreducible decomposition relative to f .*

By using 10.4 and 10.41 of [4] and 1.12 and 1.6 we obtain the

1.13. THEOREM. *Let f be a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum. Then the set of points D at which f is exactly k -to-one is dense in A if and only if A has a k -fold irreducible decomposition relative to f .*

COROLLARY. *Under the same hypothesis as the theorem, if for each x in A , $f^{-1}f(x)$ consists of at least k points, then f is k -fold irreducible if and only if A has a k -fold irreducible decomposition relative to f .*

2. Examples and applications.

2.1. *Let f be an irreducible mapping defined on a compact space A which admits a monotone light-open factorization $f = f_2 f_1$. Then f is monotone.*

PROOF. By 1.10, f_2 is irreducible and hence by Corollary 3 of 1.4, f_2 is a homeomorphism. Hence f is monotone.

In preparation for the next result we prove the following lemma.

2.2. *Let f be a mapping of a simple closed curve A onto a simple closed curve B . Then f is monotone if and only if the set D of points x in B for which $f^{-1}(x)$ is a single point is dense in B .*

PROOF. If f is monotone it is easy to see that D is dense in B . Conversely, suppose D is dense in B and let $p \in B$. Since D is dense in B , we can find sequences $p_i \rightarrow p$ and $p_i^* \rightarrow p$ such that p is between p_i and p_i^* for each i and such that all the p_i 's and p_i^* 's are in D . We may suppose that $\text{arc}(p_i p_i^*) \supset \text{arc}(p_{i+1} p_{i+1}^*)$ for each i . Further it is easy to prove that for each i , $f^{-1}(\text{arc}(p_i p_i^*))$ is an arc. Now since $p = \prod_{i=1}^{\infty} \text{arc}(p_i p_i^*)$ and since $\prod_{i=1}^{\infty} f^{-1}(\text{arc}(p_i p_i^*))$ is connected it follows that $f^{-1}(p) = f^{-1}(\prod_{i=1}^{\infty} \text{arc}(p_i p_i^*)) = \prod_{i=1}^{\infty} f^{-1}(\text{arc}(p_i p_i^*))$ is connected and hence f is monotone.

2.3. *Let f be a quasi-monotone mapping defined on a boundary curve A such that for each x in A , $f^{-1}f(x)$ consists of at least k points. Then f is k -fold irreducible if and only if f is k -to-one.*

PROOF. We prove the necessity. Let f_1 and f_2 be monotone, light open factors of f . If f is k -fold irreducible, then f_1 is an irreducible monotone mapping and hence since A is a boundary curve, it is clear that f_1 must be one-to-one and hence a homeomorphism. Then f_2 is k -fold irreducible and hence since f_2 is open f_2 must be k -to-one. The converse is obvious.

The next example follows easily from 1.13 and a similar argument to that used in 2.3.

2.4. *Let f be a quasi-monotone mapping defined on a boundary curve A . Then if A has a k -fold irreducible decomposition relative to f , f is a light-open mapping.*

The next result follows from 2.4 and a theorem of G. T. Whyburn for open mappings defined on a simple closed curve. See X, 1.2 in [3].

2.5. *Let f be a quasi-monotone mapping of A onto B where A and B are simple closed curves. Then f is k -fold irreducible (or equivalently in this case, A has a k -fold irreducible decomposition relative to f) if and only if f is topologically equivalent to the transformation $w = z^k$ defined on the circle $|z| = 1$.*

The next result is a consequence of X, 6.3 in [3] and 1.8 (a) and 1.13.

2.6. *Let f be an open mapping defined on a compact 2-manifold A . Then A has a k -fold irreducible decomposition relative to f if and only if the degree (as defined in [3]) is k .*

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