k-FOLD IRREDUCIBLE DECOMPOSITION OF A SPACE RELATIVE TO A MAPPING[1]

ROBERT H. KASRIEL

1. Introduction. Let f be a mapping of a space A onto a space B. Let $N(y, f, A, B)$, $y \in B$, be the number of points in $f^{-1}(y)$. Let k be a positive integer. According to G. T. Whyburn [2],[2] f is said to be *k-fold irreducible* provided that for $y \in B$, $N(y, f, A, B) \geqq k$, and if F is a closed proper subset of A, then for some $z \in B$, $N(z, f|F, F, B) < k$. It should be noted that if f is 1-fold irreducible, then f is *strongly irreducible* in accordance with the terminology of [3] or *irreducible* in accordance with [2].

In this paper there is introduced the notion of a k-fold irreducible decomposition of a space relative to a mapping as follows.

Definition. Let f be a mapping defined on a compact space A onto a space B. A is said to possess a *k-fold irreducible decomposition relative to* f provided that there exists a decomposition $A = \sum_{i=1}^{k} A_i$ where each A_i is a nonempty closed subset of A and the decomposition satisfies the following conditions:

(i) A_i^0, the interior (rel. to A) of A_i, is dense in A_i.

(ii) $A_i^0 \cdot A_j^0 = 0$, $i \neq j$.

(iii) $f(A_i) = B$ and $f|A_i$ is an irreducible mapping.

We further define $D(k, f, A, B)$ to be the set of all $x \in A$ for which $N(f(x), f, A, B) = k$. When there is no chance of confusion we shall write $D(k, f, A, B)$ as $D(k, f, A)$.

In 1.1 through 1.5 some definitions and results of G. T. Whyburn which will be needed are listed. In the remaining part of §1, we obtain results concerning k-fold irreducible decomposition of a space relative to certain types of mappings. One of the principal results (1.13) states that if f is a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum, then the set of points D at which f is exactly k-to-one is dense in A if and only if A has a k-fold irreducible decomposition relative to f. A corollary of this theorem is that for the same hypothesis as the theorem, if for each $x \in A$, $f^{-1}f(x)$ consists of at least k points, then f is k-fold irreducible if and only if A has a k-fold irreducible decomposition relative to f.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

In §2, some examples of k-fold irreducible decomposition are given. It is shown that if f is a quasi-monotone mapping defined on a boundary curve A such that for each $x \in A$, $f^{-1}f(x)$ consists of at least k points, then f is k-fold irreducible if and only if f is k-to-one. Also, if f is a quasi-monotone mapping defined on a boundary curve which has a k-fold irreducible decomposition relative to f, then f is a light open mapping. Finally, it is shown that for an open-mapping f defined on a compact 2-manifold A, A has a k-fold irreducible decomposition relative to f if and only if the degree of f, as defined in [**3**], is k.

For a general reference to topological terms and results used in this paper see [**3**]. All spaces are assumed to be metric.

1.1. For any set A in a metric space and any positive k, define:

$e_k(A) = \text{g.l.b.}\ [\max \delta(A_i)]$ for all decompositions $A = \sum_{i=1}^{k} A_i$ of A into k nonempty subsets A_i. Here δ stands for diameter.

For f, a mapping defined on a metric space X, define

$$e_k(x) = e_k(f^{-1}f(x)), \qquad x \in X.$$

1.2. *If the mapping f defined on X generates an upper-semi-continuous decomposition of X, the function $e_k(x)$ is upper-semi-continuous.*

Hence the theorem applies if X is compact.

1.3. *$e_k(x) = 0$ if and only if $N(f(x), f, A) \leqq k$.*

1.4. *Let f be a mapping defined on a compact space A. Suppose for each $x \in A$, $N(f(x), f, A) \geqq k$. Then f is k-fold irreducible if and only if the set of points $x \in A$ such that $N(f(x), f, A) = k$ is dense in A (i.e. $D(k, f, A)$).*

COROLLARY 1. *f is irreducible if and only if the set D of all points $x \in A$ with $x = f^{-1}f(x)$ is dense in A.*

COROLLARY 2. *If f is open, then f is k-fold irreducible if and only if f is k-to-one.*

COROLLARY 3. *If f is open, then f is irreducible if and only if f is a homeomorphism.*

1.5. *Let f be a mapping defined on a compact space A onto a space B. Then there exists a compact set $A' \subset A$ such that $f(A') = B$ and $f|A'$ is an irreducible mapping.*

1.6. THEOREM. *Let f be a mapping defined on a compact space A onto a space B. If f possesses a k-fold irreducible decomposition relative to f, then $D(k, f, A, B)$ is dense in A.*

PROOF. We first prove the following.

(*) Let U be an open set in A and let $e>0$. Then there exists an open set Q in U such that (i) if $q\in Q$, then $f^{-1}f(q)$ contains at least k points and (ii) for each $q\in Q$, $e_k(q)<e$, where $e_k(q)$ is as defined in 1.1.

Toward the end of proving this statement, let $A=\sum_{i=1}^{k} A_i$ be a k-fold irreducible decomposition relative to f and we suppose further that this decomposition is labeled so that $A_1\cdot U\neq 0$. Since A_1^0 is dense in A_1, $A_1^0\cdot U\neq 0$. Let V_1 be a nonempty open subset of $U\cdot A_1^0$ such that $0<\delta(V_1)<e$. Let $a_1\in V_1\cdot D(1, f, A_1, B)$. It is easy to prove that there exists an open set W_1 in B which contains $f(a_1)$ and such that $f^{-1}(W_1)\cdot A_1\subset V_1$. Now $f^{-1}(W_1)\cdot A_2^0$ is an open nonempty subset of A_2. Let V_2 be a nonempty subset of $f^{-1}(W_1)\cdot A_2^0$ such that $\delta(V_2)<e$. Let $a_2\in V_2\cdot D(1, f, A_2, B)$. Proceeding as before, there exists an open set W_2 in W_1 such that $W_2\supset f(a_2)$ and $f^{-1}(W_2)\cdot A_2\subset V_2$. Note that at this stage $f^{-1}(W_2)\cdot A_i$, $i=1, 2$, is such that $f[f^{-1}(W_2)\cdot A_i]=W_2$ and $\delta(f^{-1}(W_2)\cdot A_i)<e$.

It is easy to see then by induction that we are able to obtain a set W_k in B such that for $Q_i=f^{-1}(W_k)\cdot A_i$, $i=1, 2, 3, \cdots, k$, it is true that $f(Q_i)=W_k$ and $\delta(Q_i)<e$. It is now easy to verify that if we set $Q=Q_1$, then Q satisfies (i) and (ii) of our preliminary statement (*).

Let D_e be the set of all points $x\in A$ such that $e_k(x)<e$. By (ii) of *, D_e is dense in A for each $e>0$. Since e_k is upper semi-continuous it follows that the set D of all points x in A such that $e_k(x)=0$ is also dense in A. By 1.3 for each $x\in D$, $f^{-1}(x)$ contains at most k points. Next let U be an open set in A. There exists an open set Q in U such that Q satisfies (i) and (ii) of *. Since D as defined above is dense in A, there exists an $x\in Q\cdot D$. From the properties of Q and D it follows that $f^{-1}f(x)$ consists of exactly k points. Hence $x\in D(k, f, A, B)$ and $D(k, f, A, B)$ is dense in A.

The following is easy to verify.

1.7. *Let f be an open mapping of a space A onto a space B and suppose K is a compact subset of A such that $f(K)=B$ and $f|K$ is an irreducible mapping. Let K^0 be the interior (relative to A) of K. Then if $x\in K^0$, $f^{-1}f(x)\cdot K$ consists of exactly one point.*

1.8. *Let f be an open mapping defined on a compact space A onto a space B such that $D(k, f, A)$ is dense in A. Then,*

(a) *$d(f)=k$ where $d(f)=\sup\ [N(f(x), f, A)]$ for $x\in A$.*

(b) *$D(k, f, A)$ is open in A.*

(c) *f is a local homeomorphism at each x in $D(k, f, A)$.*

(d) *If F is a closed subset of A such that $f(F)=B$ and $f|F$ is irreducible, then F^0, the interior of F (rel. to A), is dense in F.*

(e) *Suppose F is a closed subset of A such that $f(F)=B$ and h is a* *positive integer such that* $1 < h \leqq k$, $D(h, f|F, F)$ *is dense in* F, *and* F^0 *is dense in* F. *Then there exists a decomposition* $F = X + Y$ *such that* X *and* Y *are closed*: X^0 *is dense in* X; Y^0 *is dense in* Y; $X^0 \cdot Y^0 = 0$; $f(X) = f(Y) = B$; $f|X$ *is irreducible*; $D(h-1, f|Y, Y)$ *is dense in* Y.

The proofs of (a), (b), and (c) follow easily from the openness of f.

PROOF. (d). Let us suppose that F^0 is not dense in F. Then there exists an open set $U \subset A$ such that $U \cdot F \neq 0$ and $U \cdot F^0 = 0$. Then either $U \cdot F \cdot D(k, f, A)$ is empty or not and we show that either case leads to a contradiction.

Suppose $U \cdot F \cdot D(k, f, A) \neq 0$. Then, by (b), $W = U \cdot D(k, f, A)$ is a nonempty open set which intersects F and since $D(1, f|F, F)$ is dense in F, there exists a point $y_1 \in W \cdot F \cdot D(1, f|F, F)$. Let $y_1 + y_2 + y_3 + \cdots + y_k = f^{-1}f(y_1)$ where we note that $y_2 + y_3 + \cdots + y_k \subset A - F$. By (c), f is a local homeomorphism at each y_i. Hence, there exist open sets U_i, $i = 1, 2, \cdots, k$, such that: $U_i \cdot U_j = 0$ for $i \neq j$, $f(U_i) = f(U_j)$, $f^{-1}f(U_i) = \sum_{i=1}^{k} U_i$, $f|U_i$ is a homeomorphism, $U_1 \subset U$, $U_i \subset A - F$ for $i = 2, 3, \cdots, k$. Since $U \cdot F^0 = 0$, there exists a $q \in U_1 \cdot (A - F)$, whence $f^{-1}f(q) \cdot F = 0$ and we have a contradiction since $f(F) = B$. Next, consider the case in which $U \cdot F \cdot D(k, f, A, B) = 0$. There exists a $y \in U \cdot F \cdot D(1, f|F, F)$. Since $D(k, f, A)$ is dense in A, there exists a sequence $y_i \to y$ such that $y_i \in U \cdot D(k, F, A)$. By the hypothesis for this case, none of the y_i's are in F. However, since $f(F) = B$ and since F is compact, there exists a sequence $x_{n_i} \to z \in F$ such that $x_{n_i} \in F$ and $f(x_{n_i}) = f(y_{n_i})$. Then since $z \in F$ and $y \in F \cdot D(1, f|F, F)$, it follows that $z = y$. Notice that if $y_{n_j} \in D(k, f, A, B)$, each point of $f^{-1}f(y_{n_j})$ is also. Hence for some j, $x_{n_j} \in U \cdot F \cdot D(k, f, A, B)$, a contradiction to the hypothesis for this case.

PROOF OF (e). There exists a closed subset $X \subset F$ such that $f(X) = B$ and such that $f|X$ is an irreducible mapping. Let Y be the closure of $F - X$. We first show that $f(Y) = B$. Suppose $f(Y) \neq B$. Then $f^{-1}(B - f(Y))$ is a nonempty open set such that $f^{-1}(B - f(Y)) \cdot Y = 0$. This leads to a contradiction, for by (d), X^0 the interior (rel. to A) of X is dense in X and further by hypothesis $D(h, f|F, F)$ is dense in F. Hence there exists an $x \in f^{-1}(B - f(Y)) \cdot X^0 \cdot D(h, f|F, F)$. By 1.7, $f^{-1}f(x) \cdot X = x$. Hence $f^{-1}f(x) \cdot (F - X)$ must contain $h - 1$ points. Thus, $f^{-1}f(x) \cdot Y \neq 0$ and we have a contradiction.

We proceed to prove that $D(h-1, f|Y, Y)$ is dense in Y. First note that Y^0, the interior (rel. to A) of Y is dense in Y. Hence, we need show only that $D(h-1, f|Y, Y)$ is dense in Y^0. Let U be an open set in Y^0. Since X^0 is dense in X and $f(X) = B$, then $f(X^0)$ is dense in B. Then, since f is open, it is easy to see that $f^{-1}f(X^0)$ is dense in A. Hence $U \cdot f^{-1}f(X^0)$ is a nonempty open subset and since

$D(h, f| F, F)$ is dense in F, it follows that there exists $x \in U \cdot f^{-1}f(X^0) \cdot D(h, f| F, F)$. By 1.7, $f^{-1}f(x) \cdot X$ is a single point, whence since $x \in D(h, f| F, F)$, $x \in D(h-1, f| Y, Y)$.

1.9. *Let f be an open mapping defined on a compact space A onto a space B. Let k be a positive integer. Then if $D(k, f, A)$ is dense in A, there exists a k-fold irreducible decomposition of A relative to f.*

PROOF. If $k=1$, the theorem is trivial. Let $k>1$. By 1.8 (e), there exists a decomposition $A=A_1+A_2$ such that A_1, A_2 are closed subsets of A, $A_1^0 \cdot A_2^0=0$, $f(A_1)=f(A_2)=B$, $f|A_1$ is irreducible, and $D(k-1, f| A_2, A_2)$ is dense in A_2. Let L be the collection of all integers m between 2 and k inclusive satisfying the condition that there exists a decomposition $A=\sum_{i=1}^m X_i$ with the following properties: (1) each X_i is closed; (2) $X_i^0 \cdot X_j^0=0$ for $i \neq j$; (3) $f(X_i)=B$ and $f|X_i$ is irreducible for $i=1, 2, 3, \cdots, m-1$; (4) $f(X_m)=B$ and $f|X_m$ satisfies the condition that $D(k-m+1, f| X_m)$ is dense in X_m; (5) X_i^0 is dense in X_i for $i=1, 2, \cdots, m$. Since $A=A_1+A_2$ satisfies the above conditions, $2 \in L$. Let $j=\max L$. Now $j=k$, for suppose $j<k$. Then there exists a decomposition $A=Y_1+Y_2+ \cdots +Y_j$ satisfying conditions (1) through (5). But $f(Y_j)=B$ and $D(k-j+1, f| Y_j)$ is dense in Y_j where it is to be noted that $k-j+1 \geqq 2$. Hence 1.8 (e) applies to $f| Y_j$ and there exists a decomposition $Y_j=Y_j^*+Y_{j+1}^*$ such that the following conditions are satisfied: $f(Y_j^*)=f(Y_{j+1}^*)=B$; $f| Y_j^*$ is irreducible; $f| Y_{j+1}^*$ is such that $D(k-j, f| Y_{j+1}^*)$ is dense in Y_{j+1}^*; $Y_j^* \cdot Y_{j+1}^*=0$; Y_{j+1}^{*0} is dense in Y_{j+1}^*. But then $A=Y_1+Y_2+ \cdots +Y_j^* + Y_{j+1}^*$ satisfies the 5 conditions and $\max L \geqq j+1$, a contradiction.

The following remark is easy to verify.

1.10. *Let f be a mapping defined on a compact space A and suppose f_1 and f_2 are any continuous factors of f. Then f is irreducible if and only if f_1 and f_2 are irreducible mappings.*

1.11. *Let f be a mapping defined on a compact space A onto a space B and let f_1 and f_2 be monotone-light factors of f. If $D(k, f, A, B)$ is a dense subset of A, then $D(1, f_1, A, f_1(A))$ is a dense subset of A and $D(k, f_2, f_1(A), B)$ is a dense subset of $f_1(A)$.*

PROOF. Because of the properties of the monotone-light factors of a mapping, $D(k, f, A, B) \subset D(1, f_1, A, f_1(A))$. Hence $D(1, f_1, A, f_1(A))$ is dense in A. Also since $D(k, f, A, B)$ is dense in A, $f_1(D(k, f, A, B))$ is dense in $f_1(A)$. But for each $z \in f_1(D(k, f, A, B))$, $N(f_2(z), f_2, f_1(A), B) = k$. Hence $f_1(D(k, f, A, B)) \subset D(k, f_2, f_1(A), B)$ and thus $D(k, f_2, f_1(A), B)$ is also dense in $f_1(A)$.

1.12. *Let f be a mapping on a compact space A onto a space B and suppose f_1, f_2 are monotone-light open factors of f. Then if $D(k, f, A, B)$ is dense in A, A has a k-fold irreducible decomposition relative to f.*

PROOF. Let $f_1(A)=X$. By 1.11, $D(1, f_1, A, X)$ is dense in A and $D(k, f_2, X, B)$ is dense in X. Hence f_1 is irreducible and, by 1.9, X has a k-fold irreducible decomposition relative to f_2. Let $X=\sum_{i=1}^{k} X_i$ be a k-fold decomposition of X relative to f_2. Let A_i be the closure of $f_1^{-1}(X_i^0)$ for $i=1, 2, \cdots, k$. We show that $\sum_{i=1}^{k} A_i$ is a k-fold decomposition of A relative to f. Since $f_1(\sum_{i=1}^{k} A_i)=X$ and f_1 is irreducible, it follows that $\sum_{i=1}^{k} A_i=A$. From the definition of A_i, $f_1^{-1}(X_i^0)$ is dense in A_i. Then since $A_1^0\supset f_1^{-1}(X_i^0)$, A_i^0 is also dense in A_i. Further, it is easy to see that $A_i^0\cdot A_j^0=0$ for $i\neq j$. Finally, we show that $f(A_i)=B$ and $f|A_i$ is an irreducible mapping. $f_1(A_i)\supset f_1f_1^{-1}(X^0)=X_i^0$. Since $f_1(A_i)$ is closed and X_i^0 is dense in X_i, $f_1(A_1)=X_i$. So $f_2f_1(A_i)=f_2(X_i)=B$. Also, $A_i^0\cdot D(1, f_1, A, X)\subset D(1, f_1|A_i, A_i, X_i)$. Then since $A_i^0\cdot D(1, f_1, A, X)$ is dense in A_i, so also is $D(1, f_1|A_i, A_i, X_i)$. Thus $f_1|A_i$ is irreducible and since $f_2|X_i$ is also, it follows that $f|A_i$ is as well.

As a corollary to the above proof we have the following

COROLLARY. *If f is a mapping defined on a compact space A with monotone light open factorization f_2f_1 and if f_1 is irreducible and the space $f_1(A)$ possesses a k-fold irreducible decomposition relative to f_2, then A possesses a k-fold irreducible decomposition relative to f.*

By using 10.4 and 10.41 of [4] and 1.12 and 1.6 we obtain the

1.13. THEOREM. *Let f be a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum. Then the set of points D at which f is exactly k-to-one is dense in A if and only if A has a k-fold irreducible decomposition relative to f.*

COROLLARY. *Under the same hypothesis as the theorem, if for each x in A, $f^{-1}f(x)$ consists of at least k points, then f is k-fold irreducible if and only if A has a k-fold irreducible decomposition relative to f.*

2. Examples and applications.

2.1. *Let f be an irreducible mapping defined on a compact space A which admits a monotone light-open factorization $f=f_2f_1$. Then f is monotone.*

PROOF. By 1.10, f_2 is irreducible and hence by Corollary 3 of 1.4, f_2 is a homeomorphism. Hence f is monotone.

In preparation for the next result we prove the following lemma.

2.2. *Let f be a mapping of a simple closed curve A onto a simple closed curve B. Then f is monotone if and only if the set D of points x in B for which $f^{-1}(x)$ is a single point is dense in B.*

PROOF. If f is monotone it is easy to see that D is dense in B. Conversely, suppose D is dense in B and let $p \in B$. Since D is dense in B, we can find sequences $p_i \rightarrow p$ and $p_i^* \rightarrow p$ such that p is between p_i and p_i^* for each i and such that all the p_i's and p_i^*'s are in D. We may suppose that arc $(p_i p p_i^*) \supset$ arc $(p_{i+1} p p_{i+1}^*)$ for each i. Further it is easy to prove that for each i, $f^{-1}(\text{arc } (p_i p p_i^*))$ is an arc. Now since $p = \prod_{i=1}^{\infty} \text{arc } (p_i p p_i^*)$ and since $\prod_{i=1}^{\infty} f^{-1}$ $(\text{arc } (p_i p p_i^*))$ is connected it follows that $f^{-1}(p) = f^{-1}$ $(\prod_{i=1}^{\infty} \text{arc } (p_i p p_i^*)) = \prod_{i=1}^{\infty} f^{-1}$ $\cdot(\text{arc } (p_i p p_i^*))$ is connected and hence f is monotone.

2.3. *Let f be a quasi-monotone mapping defined on a boundary curve A such that for each x in A, $f^{-1}f(x)$ consists of at least k points. Then f is k-fold irreducible if and only if f is k-to-one.*

PROOF. We prove the necessity. Let f_1 and f_2 be monotone, light open factors of f. If f is k-fold irreducible, then f_1 is an irreducible monotone mapping and hence since A is a boundary curve, it is clear that f_1 must be one-to-one and hence a homeomorphism. Then f_2 is k-fold irreducible and hence since f_2 is open f_2 must be k-to-one. The converse is obvious.

The next example follows easily from 1.13 and a similar argument to that used in 2.3.

2.4. *Let f be a quasi-monotone mapping defined on a boundary curve A. Then if A has a k-fold irreducible decomposition relative to f, f is a light-open mapping.*

The next result follows from 2.4 and a theorem of G. T. Whyburn for open mappings defined on a simple closed curve. See X, 1.2 in [3].

2.5. *Let f be a quasi-monotone mapping of A onto B where A and B are simple closed curves. Then f is k-fold irreducible (or equivalently in this case, A has a k-fold irreducible decomposition relative to f) if and only if f is topologically equivalent to the transformation $w = z^k$ defined on the circle $|z| = 1$.*

The next result is a consequence of X, 6.3 in [3] and 1.8 (a) and 1.13.

2.6. *Let f be an open mapping defined on a compact 2-manifold A. Then A has a k-fold irreducible decomposition relative to f if and only if the degree (as defined in [3]) is k.*

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