A THEOREM ABOUT TOPOLOGICAL n-CELLS

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1. **Introduction.** Let R^n be Euclidean n-space. We define I^n to be the set of all points $(x_1, \cdots, x_n)$ in R^n such that $0 \leqq x_j \leqq 1$ for $j = 1, \cdots, n$. Any set which is homeomorphic to I^n will be called a *topological n-cell.* The result obtained in this paper is the following:

THEOREM. *If G is a nonempty, open, connected subset of R^n, then there exists a nondecreasing sequence $E_1 \subset E_2 \subset E_3 \subset \cdots$ of topological n-cells such that $\bigcup_{m=1}^{\infty} E_m = G$.*

We point out that our topological n-cells are closed sets. It is obvious that in general G could not be represented as the union of a nondecreasing sequence of "open topological n-cells." We also point out that we do not place any type of simple connectedness restriction on G.

The sets E_m which we construct in the proof of our theorem are not only topological n-cells, but each is the union of a finite number of n-dimensional cubes. The same thing is true for the cells $E(i, j)$.

2. **Proof of the theorem.** If k is a non-negative integer and $m_1, \cdots, m_n$ are integers, we define $\sigma(k; m_1, \cdots, m_n)$ to be the set of all points $(x_1, \cdots, x_n)$ in R^n for which $m_j 2^{-k} \leqq x_j \leqq (m_j+1)2^{-k}$ for $j = 1, \cdots, n$. We define Σ to be the set of all n-cells $\sigma(k; m_1, \cdots, m_n)$ which are contained in G. Σ is countable and we arrange its members in a sequence $S_1, S_2, \cdots$.

We are going to define topological n-cells $E(i, j)$ for all positive integers i, j in such a way that:

(1) $E(i, j) \subset E(i, j+1)$ and $E(i, j) \subset E(i+1, j)$ for all i, j;

(2) $P_i = \bigcup_{j=1}^{\infty} E(i, j)$ can be expressed as the union of a finite number of members of Σ;

(3) $S_i \subset P_i$ for each i.

Once we have defined sets $E(i, j)$ having the above properties, it is easy to see that $E(1, 1) \subset E(2, 2) \subset E(3, 3) \subset \cdots$ and that $\bigcup_{m=1}^{\infty} E(m, m) = G$. Thus we can define $E_m = E(m, m)$ and obtain topological n-cells which have the desired properties.

We think of the sets $E(i, j)$ as being the elements of an infinite matrix, with i being the "row variable" and j being the "column variable." The sets $E(i, j)$ are to be defined a row at a time.

We need the following lemma.

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LEMMA 1. *Suppose that a row* $E(m, 1), E(m, 2), \cdots$ *has been defined in such a way that* (1) *and* (2) *are satisfied, and suppose that* J *is an* $(n-1)$*-cell on the boundary of* P_m. *Then there exists an integer* k *and an* $(n-1)$*-cell* J^* *such that* $J^* \subset J$ *and* $J^* \subset E(m, j)$ *for all* $j \geqq k$.

PROOF. Let $C(j) = J \cap E(m, j)$ for each positive integer j. Then each $C(j)$ is a closed subset of J. Since $J = \bigcup_{j=1}^{\infty} C(j)$ and J is of the second category with respect to itself, there exists an integer k such that $C(k)$ contains an open set relative to J. We may choose J^* to be any $(n-1)$-cell which is contained in this open set.

We begin the definitions of the sets $E(i, j)$ by defining $E(1, j) = S_1$ for each positive integer j.

Now assume that $m > 1$ and that for all $i < m$ we have defined the topological n-cells $E(i, j)$ for all j, and that these sets have been defined in such a way that (1), (2), and (3) are satisfied. We wish to define next sets $E(m, 1), E(m, 2), \cdots$. There are five cases which we consider.

Case 1. $S_m \cap P_{m-1} = \emptyset$.

Let C be the component of $G - P_{m-1}$ that contains S_m, and let J be an $(n-1)$-cell which is contained in $\bar{C} \cap P_{m-1}$. We use the lemma to obtain an integer k and an $(n-1)$-cell $J^* \subset J$ such that $J^* \subset E(m-1, j)$ for all $j \geqq k$. It is easy to see that we can unite a finite number of members of Σ to form a topological n-cell F such that: $F \subset \bar{C}$, $F \cap P_{m-1}$ is an $(n-1)$-cell which is contained in J^*, and $F \cap S_m$ is an $(n-1)$-cell. It follows that $E(m-1, j) \cup F \cup S_m$ is a topological n-cell for $j \geqq k$. We define $E(m, j) = E(m-1, j)$ for $j < k$ and $E(m, j) = E(m-1, j) \cup F \cup S_m$ for $j \geqq k$.

Case 2. P_{m-1} does not contain interior points of S_m, and $P_{m-1} \cap S_m$ contains some $(n-1)$-cell J.

We again use the lemma to obtain an integer k and an $(n-1)$-cell J^* such that $J^* \subset J \cap E(m-1, j)$ for all $j \geqq k$.

If ϵ is a positive number of the form 2^{-q}, q a positive integer, we define $S_m(\epsilon)$ to be the closure of the set obtained by subtracting from S_m all members of Σ which have edges of length ϵ and which contain points of $S_m \cap P_{m-1}$. It is obvious that for sufficiently small ϵ, $S_m(\epsilon)$ is a topological n-cell. We assume that $\delta = 2^{-q}$ is so small that $S_m(\epsilon)$ is a topological n-cell for all $\epsilon < \delta$.

It is possible to unite a finite number of members of Σ to form a topological n-cell F for which: $F \subset S_m$, $F \cup S_m(\epsilon)$ is a topological n-cell for $\epsilon < \delta$, $F \cap P_{m-1}$ is a topological $(n-1)$-cell which is contained in J^*. It follows that $E(m-1, j) \cup F \cup S_m(2^{-j}\delta)$ is a topological n-cell for $j \geqq k$. We define $E(m, j) = E(m-1, j)$ for $j < k$ and we define $E(m, j) = E(m-1, j) \cup F \cup S_m(2^{-j}\delta)$ for $j \geqq k$.

Case 3. P_{m-1} does not contain interior points of S_m, and $S_m \cap P_{m-1}$ contains only cells of dimension less than $n-1$.

We choose an $(n-1)$-cell J contained in the intersection of P_{m-1} with the boundary of the component C of $G-P_{m-1}$ which contains the interior of S_m. We also require that J be disjoint from $S_m \cap P_{m-1}$. An integer k and an $(n-1)$-cell J^* are chosen as in case 2, and sets $S_m(\epsilon)$ are defined as in case 2. We also choose δ as in case 2. It is possible to unite a finite number of members of Σ so as to form a topological n-cell F such that $F \cup S_m(\epsilon)$ is a topological n-cell for all $\epsilon < \delta$ and such that $F \cap P_{m-1}$ is an $(n-1)$-cell which is contained in J^*. We define the sets $E(m, j)$ as in case 2.

Case 4. P_{m-1} contains interior points of S_m, but does not contain all of S_m.

It is easy to see that the closure of $S_m - P_{m-1}$ can be expressed as the union of a finite set $\sigma_1, \cdots, \sigma_p$ of elements of Σ. We start with the sequence $E(m-1, 1)$, $E(m-1, 2)$, $\cdots$ and apply the construction outlined in case 2 p times to construct successively p sequences of topological n-cells. This may be done in such a way that the union of the elements of the jth sequence constructed will contain j members of the set $\sigma_1, \cdots, \sigma_p$. We take the last of the p sequences which we construct to be the sequence $E(m, 1)$, $E(m, 2)$, $\cdots$. It is easily seen that $S_m \subset \bigcup_{j=1}^{\infty} E(m, j)$.

Case 5. $S_m \subset P_{m-1}$.

We define $E(m, j) = E(m-1, j)$ for all j.

In the following corollary, $\Pi_j(G-H)$ is the jth homotopy group of $G-H$.

COROLLARY 1. *If G is a nonempty, open, connected subset of R^n, then there exists a subset H of G such that the dimension of H is less than n, $G-H$ is open and connected, and $\Pi_j(G-H)=0$ for all j.*

PROOF. Let $E_1 \subset E_2 \subset E_3 \subset \cdots$ be topological n-cells such that $\bigcup_{m=1}^{\infty} E_m = G$. We let B_m be the boundary of E_m and let $U_m = E_m - B_m$. We define $H = \bigcup_{m=1}^{\infty} \bigcap_{j=m}^{\infty} B_j$.

Since each B_j is closed and of dimension $n-1$, $\bigcap_{j=m}^{\infty} B_j$ is closed and of dimension less than n for each m. It follows from the theorem on countable unions of closed sets that H is also of dimension less than n.

It is easy to verify that $G-H = \bigcup_{m=1}^{\infty} U_m$. Since each U_m is open and connected and $U_1 \subset U_2 \subset U_3 \subset \cdots$, $G-H$ is also open and connected.

Let S be a j-sphere, and let f be a continuous function on S into $G-H$. Then $f(S)$ is compact, and it follows that $f(S) \subset U_m$ for some m. Since U_m is homeomorphic to an open n-cell in R^n, f is homotopic to a constant in U_m, and hence also in $G-H$. Thus $\Pi_j(G-H)=0$.

For the special case of open connected subsets of R^n, $n \leqq 3$, we can strengthen Corollary 1 and show that the set H can be chosen so that $G-H$ is an open topological 3-cell. Since the result is relatively trivial for $n=2$ and is completely trivial for $n=1$, we restrict our attention to the case $n=3$. We use a theorem of Moise to prove:

LEMMA 2. *If I' and I'' are closed 3-cells, J' and J'' are closed polyhedral 3-cells, I' is a subset of the interior of I'', J' is a subset of the interior of J'', and h is a piecewise linear homeomorphism of I' onto J'; then, h can be extended to a piecewise linear homeomorphism which maps I'' onto J''.*

PROOF. There exists a piecewise linear homeomorphism k of I'' onto J''. We define $A=k^{-1}h(I')$. It follows from a theorem of Moise (see [1, Theorem 1]) that there exists a piecewise linear homeomorphism ϕ of I'' onto I'' which takes A onto I'. It is easy to see that $\phi k^{-1}h$ maps I' onto I', and can thus be extended in an obvious manner to a piecewise linear homeomorphism ψ of I'' onto I''. We define $h^*=k\phi^{-1}\psi$. Then h^* is a piecewise linear homeomorphism which is an extension of h and which maps I'' onto J''.

COROLLARY 2. *If G is an open, connected subset of R^3, then there exists a subset H of G of dimension less than 3 for which there exists a piecewise linear homeomorphism of R^3 onto $G-H$.*

PROOF. It follows from our theorem (and its proof) that there exist closed polyhedral 3-cells $E_1 \subset E_2 \subset \cdots$ such that $\bigcup_{m=1}^{\infty} E_m = G$. We define U_m, B_m, and H as in Corollary 1. It is easily seen that there exist closed polyhedral 3-cells $J_m \subset U_m$ such that $\bigcup_{m=1}^{\infty} J_m = \bigcup_{m=1}^{\infty} U_m = G-H$ and such that J_m is a subset of the interior of J_{m+1} for each m.

We define I_m to be the set of all points (x_1, x_2, x_3) in R^3 for which $\max(|x_1|, |x_2|, |x_3|) \leqq m$. There exists a piecewise linear homeomorphism h_1 of I_1 onto J_1. If h_m has been defined so as to be a piecewise linear homeomorphism of I_m onto J_m, then we use Lemma 2 to extend h_m to a piecewise linear homeomorphism h_{m+1} on I_{m+1} onto J_{m+1}. If $x \in R_3$, then there exists m such that $x \in I_m$ and we define $h(x)=h_m(x)$. The function h is a piecewise linear homeomorphism of R^3 onto $G-H$.

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