EXAMPLES OF TRANSFORMATION GROUPS[1]

DEANE MONTGOMERY AND LEO ZIPPIN

1. **Introduction.** It was shown by Brouwer [1] that every finite-period homeomorphism of the ordinary plane or sphere is topologically equivalent to an orthogonal transformation. Recently R. H. Bing [2] has constructed a homeomorphism T of period two of the three-sphere, and also of three-space, which cannot be equivalent to any linear transformation. In this example, T is a "reflection" through a topological plane which is a horned-sphere of Alexander [3].

It is convenient to modify Bing's example to make an orientation-preserving homeomorphism T^* of three-space which is also of period two, and has a wildly-imbedded topological line of fixed-points; T^* is a "rotation" around the wildly-imbedded line. This transformation, too, cannot be equivalent to a linear transformation. By a theorem of Bochner [4] neither T^* nor T can be differentiable in a local coordinate system in the neighborhood of certain fixed-points.

Using T^* we shall construct a topological transformation group (C, E^4) of four-space such that C is a circle group, and such that the element of C of period two cannot be a differentiable transformation of E^4, in any coordinate system, in the neighborhood of certain of its fixed-points. Therefore the group C is not a differentiable group in any differentiable structure of E^4, and is not equivalent to a subgroup of the orthogonal group of E^4.

An example of this kind is not possible in three-space where, as the authors showed [5], a circle group of transformations is topologically equivalent to an axial rotation-group.

2. **Reflections in three-space.** Bing's example [2] may be briefly described as follows. In E^3, with coordinates x_1, x_2, x_3, let P be the plane $x_1=0$, and let R be the reflection through P:

$$R\colon (x_1, x_2, x_3) \to (-x_1, x_2, x_3).$$

There exists a dyadic family of nested anchor-rings $A_{i_1 i_2 \cdots i_n}$, $n=1, 2, \cdots; i_k=1, 2; k=1, 2, \cdots, n$, invariant under R, intersecting P in pairs of circles, and such that A_0, the intersection for all n of the union of rings of the nth stage,

$$A_0 = \bigcap_n \{\bigcup A_{i_1 \cdots i_n}\}$$

Received by the editors September 25, 1953.

[1] This paper was written while one of the authors received partial support from the Office of Naval Research.

is a family of arcs. In arbitrarily small neighborhoods of each inner point of an arc of A_0 there are paths exterior to A_0 not homotopic to a point in $(A_1 \cup A_2) - A_0$. The decomposition space $D(E^3)$ with arcs of A_0 for elements and also points of $E^3 - A_0$ as elements is homeomorphic to E^3. Denoting by D the mapping from E^3 to $D(E^3)$ the transformation defined on $D(E^3)$ by

$$T = DRD^{-1}$$

is a homeomorphism of period two with the topological plane $D(P)$ as fixed-point set. This is a wildly-imbedded set at all points of $D(P \cap A_0)$.

Finally there exists a homeomorphism D_1 of E^3 onto $D(E^3)$, showing that $D(E^3)$ is three-space.

To modify Bing's example we observe that nothing in that example is affected if we require that the centers of the circles in which the anchor-rings $A_{i_1 \cdots i_n}$ meet P shall lie on one line L. Further we suppose that this line L is the x_3-axis, $x_1 = x_2 = 0$.

Next, let R_0 denote the rotation about L through the angle π:

$$R_0(x_1, x_2, x_3) \to (-x_1, -x_2, x_3).$$

Now let anchor-rings $B_{i_1 i_2 \cdots i_n}$ be defined so as to be invariant under R_0 and to coincide with $A_{i_1 \cdots i_n}$ for $x_1 \geqq 0$. This is possible because by our choice the circle $P \cap A_{i_1 \cdots i_n}$ is invariant under R_0. Let the set of arcs B_0 be defined by

$$B_0 = \bigcap_n \{ \bigcup B_{i_1 \cdots i_n} \}$$

or equivalently by the conditions that it shall coincide with A_0 for $x_1 \geqq 0$ and shall be invariant under R_0.

It is not difficult to see that Bing's proofs apply to the present example and show that the decomposition space $D(E^3)$ with arcs of B_0 and points of $E^3 - B_0$ as elements is homeomorphic to E^3. Furthermore,

$$T^* = DR_0D^{-1}$$

defines a homeomorphism of period two in $D(E^3)$ which has $D(L)$ as fixed point set. This set is wildly-imbedded at all points of $D(L \cap B_0)$. As in Bing's example, in arbitrary neighborhoods of points of $D(B_0)$ there are paths not homotopic to a point in the set $D(B_1 \cup B_2) - D(B_0)$.

It may be of interest to remark that there is a "global" difference between the two examples. This arises from the fact that a pair of anchor-rings $A_{i_1 \cdots i_n}$, $i_n = 1, 2$, can be "pulled apart" in E^3 whereas the pair $B_{i_1 \cdots i_n}$, $i_n = 1, 2$, cannot. There is a "twist" in the relative position of these anchor-rings and there exists a pair of simple closed curves, one in each of the rings, with the algebraic linking number ± 2. That this "twist" does not affect the mobility of the space within each of the anchor-rings is clearly indicated by the fact that each arc of B_0 meets each plane $x_1 = \text{constant}$ in at most one point. This is true for the arcs of A_0—by Bing's construction. Therefore it is true for arcs of B_0 and planes $x_1 = \text{constant} \geqq 0$. Finally, it is true for all x_1 because the pair of planes $x_1 = \pm k$ is invariant under R_0.

3. **A lemma.** We prove a lemma which will be useful in the example to follow: In that example, K will be a closed 3-cell.

LEMMA. *Let K be a compact space and I a closed unit interval. Let T_0 and T_1 be two homeomorphisms of K into itself. Form the decomposition space M_0 from $K \times I$ by identifying the point-pairs $(x, 0)$ and $(T_0(x), 1)$. Similarly, form M_1 by identifying $(x, 0)$ and $(T_1(x), 1)$. Then if T_0 and T_1 are isotopic over K, M_0 and M_1 will be homeomorphic.*

Since T_0 and T_1 are isotopic, there is a family of homeomorphisms $T(x, t)$ of K onto K simultaneously continuous in $x \in K$ and $t \in [0, 1]$ such that

$$T(x, 0) = T_0(x), \; T(x, 1) = T_1(x).$$

There is a homeomorphism H of $K \times I$ onto itself defined by

$$H: (x, t) \to (T(T_0^{-1}(x), t), t).$$

Note that the pair $(x, 0)$ and $(T_0(x), 1)$ pertinent to the space M_0 is carried into the pair $(x, 0)$ and $(T_1(x), 1)$ belonging to M_1. Then it is clear that H maps M_0 upon M_1 in a continuous fashion. The map is one-one from the fact that H is a homeomorphism of $K \times t$ onto itself for each $t \in [0, 1]$. Thus H determines a homeomorphism of M_0 upon M_1.

4. **An example in four-space.** To construct an example of a circle group C acting on E^4 in an essentially nondifferentiable way we first consider C acting linearly on an E^4 with coordinates x_1, x_2, x_3, x_4 as follows:

$$\begin{aligned} x_1' &= x_1 \cos 2\pi t - x_2 \sin 2\pi t, & x_2' &= x_1 \sin 2\pi t + x_2 \cos 2\pi t, \\ x_3' &= x_3 \cos 4\pi t - x_4 \sin 4\pi t, & x_4' &= x_3 \sin 4\pi t + x_4 \cos 4\pi t. \end{aligned}$$

Under this group all points have period 1, except that points $(0, 0, x_3, x_4)$ have period 1/2, and the origin is fixed for all t.

Let E^3 be the subspace $x_4 = 0$, let P be the plane $x_1 = x_4 = 0$ and let

L be the x_3-axis. The group element $t=1/2\in C$ takes E^3 into itself, P into itself, and leaves L fixed. Clearly $t=1/2$ is the rotation R_0 of the preceding example. In E^3 construct the set B_0 as before—except that B_0 may be supposed interior to a solid sphere K invariant under $t=1/2$ and not including the origin.

When C operates on E^4 the set B_0 sweeps out a set $C(B_0)$ whose components are Möbius bands determined by the individual arcs of B_0 for each of which one and only one point is on L. Consider the decomposition space $D(E^4)$ whose elements are the points of $E^4-C(B_0)$ and also each set of the form cJ where $c\in C$ and J is a constituent arc of B_0.

The elements of $D(E^4)$ are permuted among themselves by C, and C is a topological transformation group of $D(E^4)$. We shall prove that $D(E^4)$ is homeomorphic to E^4, and that C is not differentiable in any differentiable structure of $D(E^4)$.

The space E^4 is the union of two sets invariant under C, namely $C(K)$ and $E^4-C(K)$. Similarly $D(E^4)$ is the union of $D(C(K))$ and $D(E^4)-D(C(K))$. Now D is a homeomorphism on the closure of $E^4-C(K)$ and on the boundary of $C(K)$. We want to extend this to a homeomorphism of $C(K)$ and $D(C(K))$.

Since B_0 belongs to K, the decomposition space of K whose elements are points of $K-B_0$ and arcs of B_0 is a subset $D(K)$ of $D(E^4)$. Since B_0 is interior to K, D is a homeomorphism on the boundary of K to the boundary of $D(K)$. We know that this homeomorphism can be extended to a homeomorphism D_1: $K\rightarrow D(K)$.

Let T^* denote the homeomorphism of period two of $D(K)$ onto itself defined in the preceding example by $T^*=DR_0D^{-1}$. Let R_1 of period two on K be defined by

$$R_1 = D_1^{-1}T^*D_1.$$

Because D_1 coincides with D on the boundary of K, R_1 coincides with R_0 on the boundary of K.

Therefore, by a theorem of Alexander [6] (a closely related result has been proved by Veblen [7]) R_0 and R_1 are isotopic through a family $R(x, t)$ of homeomorphisms of K each of which coincides with R_0 on the boundary of K.

Therefore, by the lemma above, the decomposition space M_0 obtained from $K\times I$ by identifying $(x, 0)$ with $(R_0(x), 1)$ for every $t\in K$ is homeomorphic to M_1, the decomposition space of $K\times I$ which results from identifying the pairs $(x, 0)$ and $(R_1(x), 1)$. But the space M_0 is clearly homeomorphic to the set $C(K)$. There remains to show that M_1 is homeomorphic to $D(C(K))$.

It is clear that $D(C(K))$ is a fibre-space with fibre homeomorphic to $D(K)$ and base a simple closed curve. In fact $D(C(K))$ is identical with the subset $C(D(K))$ of $D(E^4)$. This set is homeomorphic to the set M_2 obtained from the product $D(K)\times I$ by identifying the pairs $(y, 0)$ and $(T^*y, 1)$, for every $y \in D(K)$. Recalling the homeomorphism D_1 and the relation of R_1 to T^*, it can be seen that M_1 is homeomorphic to M_2. This completes the proof that E^4 is homeomorphic to $D(E^4)$.

The group C acts in a natural way on $D(E^4)$ since it permutes the subsets of E^4 which constitute the elements of the decomposition space, and C is a topological transformation group of $D(E^4)$.

To show that the group C cannot be everywhere differentiable in any differentiable structure for $D(E^4)$ it is sufficient to consider the element $t=1/2 \in C$ acting on a neighborhood of a point of $D(B_0)$. The transformation $t=1/2$ leaves fixed all points of $D(C(L))$, a two-dimensional cone with vertex at the "origin"—homeomorphic to $C(L) \subset E^4$ where the origin is a fixed point under all of C. In particular $t=1/2$ leaves fixed all points of $D(B_0)\subset D(L)$.

Let b be a point of $D(B_0)$. We know from the construction of B_0 that there is a neighborhood V of b in $D(K)$ such that there are arbitrarily small neighborhoods of b in which there exist paths exterior to $D(L)$ not homotopic to a point in $V-D(B_0)$. Let us consider such a neighborhood U in $D(K)$ and path Q in $U-D(L)$ not homotopic to a point in $V-D(B_0)$.

Let S be an arbitrary segment of elements of C containing the identity. Then $S(U)$ is a neighborhood of b in $D(E^4)$. If Q were homotopic to a point in $S(U)-S(D(L))$, the family of paths deforming it to a point could be projected along orbits under S to become a deformation family for Q in $U-D(L)$. This contradicts the choice of Q.

We see in this way that the cone $C(D(L))$ is not tamely-imbedded in $D(E^4)$ at any point of $D(B_0)$ and cannot be equivalent to a linear manifold at such points. But, by Bochner's theorem, if the element $t=1/2$ of C were a differentiable transformation at the fixed-point b, this transformation would be linear in appropriate coordinates and the fixed-point set would be linear, at least locally. This concludes the proof.

It may be of interest to note that it is not difficult to obtain, from Bing's original example, an example similar to the one above but with a "plane" of points fixed under all elements of the circle group C. However, the authors do not know whether the resulting space is E^4 nor whether an example with a plane of points fixed under all elements of C is possible for E^4.

BIBLIOGRAPHY

1. Brouwer, *Über die periodischen Transformationen der Kreisscheibe und der Kugelfläche*, Math. Ann. vol. 80 (1919) pp. 39–41. See also Kerékjártó, *Vorlesungen über Topologie* I, Berlin, Springer, 1923 and Eilenberg, *Sur les transformations périodiques de la surface de sphère*, Fund. Math. vol. 22 (1933) pp. 28–41.

2. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. vol. 56 (1952) pp. 354–362.

3. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 8–10.

4. Bochner, *Compact groups of differentiable transformations*, Ann. of Math. vol. 46 (1945) pp. 372–381.

5. Montgomery and Zippin, *Periodic one-parameter groups in three-space*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 24–36.

6. Alexander, *On the deformation of an n-cell*, Proc. Nat. Acad. Sci. U.S.A. vol. 9 (1923) pp. 406–407.

7. Veblen, *On the deformation of an n-cell*, Proc. Nat. Acad. Sci. U.S.A. vol. 3 (1917) pp. 654–656.

INSTITUTE FOR ADVANCED STUDY AND
QUEENS COLLEGE