ON INTERVALS OF PRESCRIBED LENGTHS

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The present paper is an outgrowth of a conversation with Maurice Sion. In this conversation he asked whether the closed unit interval always contains a set of Lebesgue measure zero which cannot be embraced by a sequence of open intervals of prescribed lengths adding up to 1. I posed the corresponding question for the unit open interval. Answers to these questions are given by Theorems 3 and 6 below.

Let us agree: ω is the set of non-negative integers; a *sequence* is a function whose domain is ω; S *embraces* A if and only if S is such a sequence of sets that

$$A \subset \bigcup_{n\in\omega} S_n;$$

$\mathcal{L}$ is (outer) Lebesgue measure; J is a λ *file* if and only if J is such a sequence of open intervals and λ is such a sequence of non-negative real numbers that

$$\mathcal{L}(J_n) = \lambda_n \qquad \text{whenever } n \in \omega;$$

$$I = \mathrm{E}t(0 < t < 1);$$

$$I' = \mathrm{E}t(0 \leqq t \leqq 1);$$

S is *separate* if and only if S is such a sequence of sets that

$$S_m \cap S_n = 0 \qquad \text{whenever } m \in \omega,\ n \in \omega,\ m \neq n;$$

subsum λ is the set of numbers of the form

$$\sum_{n\in\alpha} \lambda_n$$

where $\alpha \subset \omega$.

Lemma 1 below is of considerable use to us here and is easily seen with the aid of E. M. Beesley and A. P. Morse, *ϕ-Cantorian functions and their convex moduli*, Duke Math. J. vol. 12 (1945) p. 608, Theorem 7.1.

1. LEMMA. *If λ is a sequence of non-negative real numbers,*

$$\sum_{n\in\omega} \lambda_n < 1,$$

then I contains such a set A of Lebesgue measure zero that no λ file embraces A.

2. THEOREM. *If λ is a sequence of non-negative real numbers,*

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$$\sum_{n \in \omega} \lambda_n = 1,$$

then I contains such a set A of Lebesgue measure zero that

$$J \textit{ is separate and } \bigcup_{n \in \omega} J_n \subset I$$

whenever J is a λ file which embraces A.

PROOF. Let F be the family of such sequences μ that for some $k \in \omega$:

$$\mu_n = \lambda_n \qquad \text{whenever } n \in \omega \text{ and } n \neq k;$$

$$\mu_k \text{ is rational and } 0 < \mu_k < \lambda_k.$$

Clearly F is countable. Furthermore each $\mu \in F$ is such a sequence of non-negative real numbers that

$$\sum_{n \in \omega} \mu_n < 1.$$

Consequently Lemma 1 and the principle of choice enable us to associate with each $\mu \in F$ such a set B_μ that:

$$B_\mu \subset I, \qquad \mathcal{L}(B_\mu) = 0,$$

$$\text{no } \mu \text{ file embraces } B_\mu. \tag{1}$$

Now let

$$A = \bigcup_{\mu \in F} B_\mu.$$

Clearly

$$A \subset I, \qquad \mathcal{L}(A) = 0;$$

and completion of the proof is tantamount to verification of the

STATEMENT. *If J is any λ file which embraces A then*

$$J \textit{ is separate and } \bigcup_{n \in \omega} J_n \subset I.$$

PROOF. Suppose the contrary and so choose $p \in \omega$ that either:

$$0 \neq J_p \cap J_n \qquad \text{and } \sup J_p \leqq \sup J_n$$

for some $n \in \omega$ with $n \neq p$; or

$$J_p \not\subset I.$$

In either event it is clearly possible to so ascertain a sequence K of open intervals that:

$$K_n = J_n \qquad \text{whenever } n \in \omega \text{ with } n \neq p;$$

$$K_p \subset J_p;$$

$$\mathcal{L}(K_p) \qquad \text{is rational and } 0 < \mathcal{L}(K_p) < \mathcal{L}(J_p);$$

$$I \cap \bigcup_{n \in \omega} K_n = I \cap \bigcup_{n \in \omega} J_n;$$
$$K \text{ embraces } A.$$

Letting ν be the sequence for which

$$\nu_n = \mathcal{L}(K_n) \qquad \text{whenever } n \in \omega$$

we see at once that

$$\nu \in F, \quad K \text{ is a } \nu \text{ file which embraces } A.$$

Hence, in contradiction to (1),

$$\nu \in F, \quad K \text{ is a } \nu \text{ file which embraces } B_\nu.$$

3. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$\sum_{n \in \omega} \lambda_n = 1,$$

then I' contains such a set A' of Lebesgue measure zero that no λ file embraces A'.

PROOF. Choose A in accordance with Theorem 2 and form A' by adjoining 0 to A.

4. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$0 < \lambda_0 < 1,$$
$$\sum_{n \in \omega} \lambda_n = 1,$$
$$H = \text{subsum } \lambda,$$
$$\mathcal{L}(H) = 0,$$

then I contains such a set B of Lebesgue measure zero that no λ file embraces B.

PROOF. Choose A in accordance with Theorem 2 and let

$$B = A \cup (H \cap I).$$

Clearly

$$\mathcal{L}(B) = 0 \quad \text{and} \quad B \subset I;$$

that no λ file embraces B is a consequence of the

STATEMENT. *If J is a λ file which embraces A then J does not embrace $H \cap I$.*

PROOF. Since J embraces A we infer from Theorem 2 that

(1) $$J \text{ is separate and } \bigcup_{n \in \omega} J_n \subset I.$$

Because of this and the fact that

$$0 < \lambda_0 < 1$$

we choose such an end point ξ of J_0 that

$$\xi \in I.$$

Since J is separate we are sure

(2) $$\xi \notin \bigcup_{n \in \omega} J_n.$$

Now let

$$a = \mathrm{E}t(t < \xi), \qquad b = \mathrm{E}t(\xi < t),$$
$$\alpha = \mathrm{E}n \in \omega[J_n \subset a], \qquad \beta = \mathrm{E}n \in \omega[J_n \subset b].$$

Because of (2) it is evident that

$$\alpha \cup \beta = \omega$$

and, because of (1), it follows that

$$\begin{aligned} 0 &\geqq \textstyle\sum_{n \in \alpha} \mathcal{L}(J_n) - \xi \\ &\geqq \textstyle\sum_{n \in \alpha} \mathcal{L}(J_n) - \xi + \sum_{n \in \beta} \mathcal{L}(J_n) - 1 + \xi \\ &\geqq \textstyle\sum_{n \in \omega} \mathcal{L}(J_n) - 1 \\ &= \textstyle\sum_{n \in \omega} \lambda_n - 1 \\ &= 0. \end{aligned}$$

Consequently

$$\xi = \textstyle\sum_{n \in \alpha} \mathcal{L}(J_n) = \sum_{n \in \alpha} \lambda_n$$

and thus

$$\xi \in H \cap I.$$

Reference to (2) now convinces us that J does not embrace $H \cap I$.

5. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$\begin{aligned} &\textstyle\sum_{n \in \omega} \lambda_n = 1, \\ &H = \text{subsum } \lambda, \\ &\mathcal{L}(H) > 0, \\ &A \subset I, \\ &\mathcal{L}(A) = 0, \end{aligned}$$

then there is a λ file which embraces A.

PROOF. Let us agree herein that

$$T^*(W)$$

is the image under the function T of the set W. Let us also agree herein that R is the set of real finite numbers.

Let C be the set of such functions T on R that for some $n \in \omega$

$$T(x) = x + \lambda_n \qquad \text{whenever } x \in R.$$

Let G be the smallest transformation group which contains C. Since C is a countable set of translations of R into itself it follows that G is also a countable set of translations of R into itself.

We let

$$B = \bigcup_{T \in G} T^*(A)$$

and divide the remainder of the proof into seven parts. Of these Part I is obvious, Part II is an immediate consequence of the countability of G, Parts III and IV are, no doubt, special cases of well known general theorems, and Part VII is the conclusion desired.

PART I. $A \subset I \cap B$.

PART II. $\mathcal{L}(B) = 0$.

PART III. *If* $S \in G$ *then* $S^*(B) \subset B$.

PROOF.

$$\begin{aligned} S^*(B) &= S^*\left(\bigcup_{T \in G} T^*(A)\right) \\ &= \bigcup_{T \in G} S^*(T^*(A)) \\ &\subset B. \end{aligned}$$

PART IV. *If* $T \in G$ *then* $T^*(R \sim B) = R \sim B$.

PROOF. Let P be the inverse of T. From Part III we learn

$$P^*(B) \subset B.$$

Hence

$$B = T^*(P^*(B)) \subset T^*(B)$$

and re-use of Part III assures us

$$T^*(B) = B.$$

Accordingly

$$T^*(R \sim B) = T^*(R) \sim T^*(B) = R \sim B.$$

PART V. *If* $n \in \omega$, $x \in R \sim B$, $r^2 = 1$, *then*

$$x + r \cdot \lambda_n \in R \sim B.$$

PROOF. Use Part IV.

Easily deduced from Part V is

PART VI. *If γ is a finite subset of ω, $x\in R\sim B$, $r^2=1$, then*

$$x + r\cdot \sum_{m\in\gamma} \lambda_m \in R \sim B.$$

PART VII. *Some λ file embraces A.*

PROOF. Since $\mathcal{L}(B)=0$ we are sure

$$\mathcal{L}(H \sim B) = \mathcal{L}(H) > 0,$$

and we so choose ξ that

$$\xi \in H \sim B,$$

and then so choose $\alpha\subset\omega$ that

$$\xi = \sum_{n\in\alpha} \lambda_n.$$

Let

$$\beta = \omega \sim \alpha$$

and note

$$1 - \xi = \sum_{n\in\beta} \lambda_n.$$

For each $n\in\omega$ let:

$$M_n = \mathrm{E}m \in \alpha[0 \leqq m < n];$$
$$N_n = \mathrm{E}m \in \beta[0 \leqq m < n].$$

Let J be such a sequence that:

$$J_n = \mathrm{E}t[\xi - \lambda_n < t + \sum_{m\in M_n} \lambda_m < \xi]$$

whenever $n\in\alpha$;

$$J_n = \mathrm{E}t[\xi < t - \sum_{m\in N_n} \lambda_m < \xi + \lambda_n]$$

whenever $n\in\beta$.

Clearly J is a λ file.

If $0<x\leqq\xi$ and

$$p = \sup \mathrm{E}_{n\in\omega}[x + \sum_{m\in M_n} \lambda_m \leqq \xi]$$

then $p\in\alpha$ and either

$$x \in J_p \quad \text{or} \quad x = \xi - \sum_{m\in M_p} \lambda_m.$$

Similarly if $\xi\leqq x<1$ and

$$q = \sup \mathrm{E}_{n\in\omega}[\xi \leqq x - \sum_{m\in N_n} \lambda_m]$$

then $q\in\beta$ and either

$$x \in J_q \quad \text{or} \quad x = \xi + \textstyle\sum_{m \in N_q} \lambda_m.$$

From the last two paragraphs, from Part VI, and from the fact that

$$\xi \in H \sim B \subset R \sim B$$

it follows that

$$I \subset R \sim B \cup \textstyle\bigcup_{n \in \omega} J_n.$$

According to this and Part I

$$A \subset I \cap B \subset \textstyle\bigcup_{n \in \omega} J_n$$

and

$$J \text{ embraces } A.$$

Since it has already been noted that J is a λ file the proof of Part VII is now complete.

6. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$0 < \lambda_0 < 1,$$
$$\textstyle\sum_{n \in \omega} \lambda_n = 1,$$
$$H = \text{subsum } \lambda,$$

then a necessary and sufficient condition that I contains such a set A of Lebesgue measure zero that no λ file embraces A is that

$$\mathcal{L}(H) = 0.$$

PROOF. Use Theorems 4 and 5.

7. REMARKS. If λ is such a sequence that

$$\lambda_n = 2^{-1} \cdot 2^{-n} \qquad \text{whenever } n \in \omega$$

then

$$\textstyle\sum_{n \in \omega} \lambda_n = 1$$

and subsum $\lambda = I'$.

Less obvious is the fact that if λ is such a sequence that

$$\lambda_n = 3 \cdot 10^{-1} \cdot (2^{-n} + 4^{-n}) \qquad \text{whenever } n \in \omega$$

then

$$\textstyle\sum_{n \in \omega} \lambda_n = 1$$

and subsum λ is a nondense perfect set of Lebesgue measure 3/5.

Finally if λ is such a sequence that

$$\lambda_n = 2\cdot 3^{-1}\cdot 3^{-n} \qquad \text{whenever } n \in \omega$$

then subsum λ is the Cantor ternary set of Lebesgue measure zero.

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BOUNDS FOR THE MODULI OF THE ZEROS OF A POLYNOMIAL

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1. **Introduction.** Let $f(z)=a_0+a_1z+\cdots+a_nz^n$, $a_0\neq 0$, $a_n\neq 0$. Following the notation of Ostrowski [1], we define the Newton diagram of $f(z)$ as the broken line D in the (x, y)-plane with the following properties:

(1) D extends from the point $x=0$, $y=-\log|a_0|$ to the point $x=n$, $y=-\log|a_n|$;

(2) $y_\nu\leqq(y_{\nu-1}+y_{\nu+1})/2$ $(\nu=1, 2, \cdots, n-1)$, where y_ν is the ordinate of D for $x=\nu$;

(3) The points $(\nu, -\log|a_\nu|)$ $(\nu=0, 1, \cdots, n)$ all lie on or above D;

(4) If (ν, y_ν) is a corner point of D, i.e., $y_\nu<(y_{\nu-1}+y_{\nu+1})/2$, then $y_\nu=-\log|a_\nu|$.

Values of ν for which $(\nu, -\log|a_\nu|)$ is a corner point or an end point of D are called principal indices of $f(z)$. Let $y_\nu=-\log T_\nu$ so $T_\nu=|a_\nu|$ when ν is a principal index, and let $R_\nu=T_{\nu-1}/T_\nu$. The slope of D in the interval $(\nu-1, \nu)$ is $\log R_\nu$.

Let the zeros of $f(z)$ be denoted by $z_1, z_2, \cdots, z_n$ with $0<|z_1|\leqq|z_2|\leqq\cdots\leqq|z_n|$, and let b_k^n and B_k^n $(k=1, 2, \cdots, n)$ be the minimum and maximum respectively of $|z_k|/R_k$ for all polynomials of degree n which do not vanish at the origin. Ostrowski [1] has shown, by considering the polynomial $z^nf(1/z)$, that

$$B_k^n = 1/b_{n-k+1}^n. \tag{1}$$

Ostrowski proved that b_1^n is the positive root of the equation $1=x+x^2+\cdots+x^n$, that b_n^n is equal to $1/n$, and that

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