

ON INTERVALS OF PRESCRIBED LENGTHS

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The present paper is an outgrowth of a conversation with Maurice Sion. In this conversation he asked whether the closed unit interval always contains a set of Lebesgue measure zero which cannot be embraced by a sequence of open intervals of prescribed lengths adding up to 1. I posed the corresponding question for the unit open interval. Answers to these questions are given by Theorems 3 and 6 below.

Let us agree: ω is the set of non-negative integers; a *sequence* is a function whose domain is ω ; S *embraces* A if and only if S is such a sequence of sets that

$$A \subset \bigcup_{n \in \omega} S_n;$$

\mathcal{L} is (outer) Lebesgue measure; J is a λ *file* if and only if J is such a sequence of open intervals and λ is such a sequence of non-negative real numbers that

$$\begin{aligned} \mathcal{L}(J_n) &= \lambda_n && \text{whenever } n \in \omega; \\ I &= \text{Et}(0 < t < 1); \\ I' &= \text{Et}(0 \leq t \leq 1); \end{aligned}$$

S is *separate* if and only if S is such a sequence of sets that

$$S_m \cap S_n = \emptyset \quad \text{whenever } m \in \omega, n \in \omega, m \neq n;$$

subsum λ is the set of numbers of the form

$$\sum_{n \in \alpha} \lambda_n$$

where $\alpha \subset \omega$.

Lemma 1 below is of considerable use to us here and is easily seen with the aid of E. M. Beesley and A. P. Morse, *ϕ -Cantorian functions and their convex moduli*, Duke Math. J. vol. 12 (1945) p. 608, Theorem 7.1.

1. LEMMA. *If λ is a sequence of non-negative real numbers,*

$$\sum_{n \in \omega} \lambda_n < 1,$$

then I contains such a set A of Lebesgue measure zero that no λ file embraces A .

2. THEOREM, *If λ is a sequence of non-negative real numbers,*

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$$\sum_{n \in \omega} \lambda_n = 1,$$

then I contains such a set A of Lebesgue measure zero that

$$J \text{ is separate and } \bigcup_{n \in \omega} J_n \subset I$$

whenever J is a λ file which embraces A .

PROOF. Let F be the family of such sequences μ that for some $k \in \omega$:

$$\begin{aligned} \mu_n &= \lambda_n && \text{whenever } n \in \omega \text{ and } n \neq k; \\ \mu_k &\text{ is rational and } 0 < \mu_k < \lambda_k. \end{aligned}$$

Clearly F is countable. Furthermore each $\mu \in F$ is such a sequence of non-negative real numbers that

$$\sum_{n \in \omega} \mu_n < 1.$$

Consequently Lemma 1 and the principle of choice enable us to associate with each $\mu \in F$ such a set B_μ that:

$$(1) \quad \begin{aligned} B_\mu &\subset I, && \mathcal{L}(B_\mu) = 0, \\ &\text{no } \mu \text{ file embraces } B_\mu. \end{aligned}$$

Now let

$$A = \bigcup_{\mu \in F} B_\mu.$$

Clearly

$$A \subset I, \quad \mathcal{L}(A) = 0;$$

and completion of the proof is tantamount to verification of the

STATEMENT. *If J is any λ file which embraces A then*

$$J \text{ is separate and } \bigcup_{n \in \omega} J_n \subset I.$$

PROOF. Suppose the contrary and so choose $p \in \omega$ that either:

$$0 \neq J_p \cap J_n \quad \text{and } \sup J_p \leq \sup J_n$$

for some $n \in \omega$ with $n \neq p$; or

$$J_p \not\subset I.$$

In either event it is clearly possible to so ascertain a sequence K of open intervals that:

$$\begin{aligned} K_n &= J_n && \text{whenever } n \in \omega \text{ with } n \neq p; \\ K_p &\subset J_p; \\ \mathcal{L}(K_p) &\text{ is rational and } 0 < \mathcal{L}(K_p) < \mathcal{L}(J_p); \end{aligned}$$

$$I \cap \bigcup_{n \in \omega} K_n = I \cap \bigcup_{n \in \omega} J_n;$$

K embraces A .

Letting ν be the sequence for which

$$\nu_n = \mathcal{L}(K_n) \quad \text{whenever } n \in \omega$$

we see at once that

$$\nu \in F, \quad K \text{ is a } \nu \text{ file which embraces } A.$$

Hence, in contradiction to (1),

$$\nu \in F, \quad K \text{ is a } \nu \text{ file which embraces } B_\nu.$$

3. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$\sum_{n \in \omega} \lambda_n = 1,$$

then I' contains such a set A' of Lebesgue measure zero that no λ file embraces A' .

PROOF. Choose A in accordance with Theorem 2 and form A' by adjoining 0 to A .

4. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$\begin{aligned} 0 < \lambda_0 < 1, \\ \sum_{n \in \omega} \lambda_n &= 1, \\ H &= \text{subsum } \lambda, \\ \mathcal{L}(H) &= 0, \end{aligned}$$

then I contains such a set B of Lebesgue measure zero that no λ file embraces B .

PROOF. Choose A in accordance with Theorem 2 and let

$$B = A \cup (H \cap I).$$

Clearly

$$\mathcal{L}(B) = 0 \quad \text{and} \quad B \subset I;$$

that no λ file embraces B is a consequence of the

STATEMENT. *If J is a λ file which embraces A then J does not embrace $H \cap I$.*

PROOF. Since J embraces A we infer from Theorem 2 that

$$(1) \quad J \text{ is separate and } \bigcup_{n \in \omega} J_n \subset I.$$

Because of this and the fact that

$$0 < \lambda_0 < 1$$

we choose such an end point ξ of J_0 that

$$\xi \in I.$$

Since J is separate we are sure

$$(2) \quad \xi \notin \bigcup_{n \in \omega} J_n.$$

Now let

$$\begin{aligned} a &= \text{El}(t < \xi), & b &= \text{El}(\xi < t), \\ \alpha &= \text{En} \in \omega [J_n \subset a], & \beta &= \text{En} \in \omega [J_n \subset b]. \end{aligned}$$

Because of (2) it is evident that

$$\alpha \cup \beta = \omega$$

and, because of (1), it follows that

$$\begin{aligned} 0 &\geq \sum_{n \in \alpha} \mathcal{L}(J_n) - \xi \\ &\geq \sum_{n \in \alpha} \mathcal{L}(J_n) - \xi + \sum_{n \in \beta} \mathcal{L}(J_n) - 1 + \xi \\ &\geq \sum_{n \in \omega} \mathcal{L}(J_n) - 1 \\ &= \sum_{n \in \omega} \lambda_n - 1 \\ &= 0. \end{aligned}$$

Consequently

$$\xi = \sum_{n \in \alpha} \mathcal{L}(J_n) = \sum_{n \in \alpha} \lambda_n$$

and thus

$$\xi \in H \cap I.$$

Reference to (2) now convinces us that J does not embrace $H \cap I$.

5. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$\sum_{n \in \omega} \lambda_n = 1,$$

$$H = \text{subsum } \lambda,$$

$$\mathcal{L}(H) > 0,$$

$$A \subset I,$$

$$\mathcal{L}(A) = 0,$$

then there is a λ file which embraces A .

PROOF. Let us agree herein that

$$T^*(W)$$

is the image under the function T of the set W . Let us also agree herein that R is the set of real finite numbers.

Let C be the set of such functions T on R that for some $n \in \omega$

$$T(x) = x + \lambda_n \quad \text{whenever } x \in R.$$

Let G be the smallest transformation group which contains C . Since C is a countable set of translations of R into itself it follows that G is also a countable set of translations of R into itself.

We let

$$B = \bigcup_{T \in G} T^*(A)$$

and divide the remainder of the proof into seven parts. Of these Part I is obvious, Part II is an immediate consequence of the countability of G , Parts III and IV are, no doubt, special cases of well known general theorems, and Part VII is the conclusion desired.

PART I. $A \subset I \cap B$.

PART II. $\mathcal{L}(B) = 0$.

PART III. If $S \in G$ then $S^*(B) \subset B$.

PROOF.

$$\begin{aligned} S^*(B) &= S^*(\bigcup_{T \in G} T^*(A)) \\ &= \bigcup_{T \in G} S^*(T^*(A)) \\ &\subset B. \end{aligned}$$

PART IV. If $T \in G$ then $T^*(R \sim B) = R \sim B$.

PROOF. Let P be the inverse of T . From Part III we learn

$$P^*(B) \subset B.$$

Hence

$$B = T^*(P^*(B)) \subset T^*(B)$$

and re-use of Part III assures us

$$T^*(B) = B.$$

Accordingly

$$T^*(R \sim B) = T^*(R) \sim T^*(B) = R \sim B.$$

PART V. If $n \in \omega$, $x \in R \sim B$, $r^2 = 1$, then

$$x + r \cdot \lambda_n \in R \sim B.$$

PROOF. Use Part IV.

Easily deduced from Part V is

PART VI. If γ is a finite subset of ω , $x \in R \sim B$, $r^2 = 1$, then

$$x + r \cdot \sum_{m \in \gamma} \lambda_m \in R \sim B.$$

PART VII. Some λ file embraces A .

PROOF. Since $\mathcal{L}(B) = 0$ we are sure

$$\mathcal{L}(H \sim B) = \mathcal{L}(H) > 0,$$

and we so choose ξ that

$$\xi \in H \sim B,$$

and then so choose $\alpha \subset \omega$ that

$$\xi = \sum_{n \in \alpha} \lambda_n.$$

Let

$$\beta = \omega \sim \alpha$$

and note

$$1 - \xi = \sum_{n \in \beta} \lambda_n.$$

For each $n \in \omega$ let:

$$M_n = \{m \in \alpha \mid 0 \leq m < n\};$$

$$N_n = \{m \in \beta \mid 0 \leq m < n\}.$$

Let J be such a sequence that:

$$J_n = \{t \mid \xi - \lambda_n < t + \sum_{m \in M_n} \lambda_m < \xi\}$$

whenever $n \in \alpha$;

$$J_n = \{t \mid \xi < t - \sum_{m \in N_n} \lambda_m < \xi + \lambda_n\}$$

whenever $n \in \beta$.

Clearly J is a λ file.

If $0 < x \leq \xi$ and

$$p = \sup_{n \in \omega} \{x + \sum_{m \in M_n} \lambda_m \leq \xi\}$$

then $p \in \alpha$ and either

$$x \in J_p \quad \text{or} \quad x = \xi - \sum_{m \in M_p} \lambda_m.$$

Similarly if $\xi \leq x < 1$ and

$$q = \sup_{n \in \omega} \{\xi \leq x - \sum_{m \in N_n} \lambda_m\}$$

then $q \in \beta$ and either

$$x \in J_q \text{ or } x = \xi + \sum_{m \in N_q} \lambda_m.$$

From the last two paragraphs, from Part VI, and from the fact that

$$\xi \in H \sim B \subset R \sim B$$

it follows that

$$I \subset R \sim B \cup \bigcup_{n \in \omega} J_n.$$

According to this and Part I

$$A \subset I \cap B \subset \bigcup_{n \in \omega} J_n$$

and

J embraces A .

Since it has already been noted that J is a λ file the proof of Part VII is now complete.

6. THEOREM. *If λ is a sequence of non-negative real numbers,*

$$0 < \lambda_0 < 1,$$

$$\sum_{n \in \omega} \lambda_n = 1,$$

$$H = \text{subsum } \lambda,$$

then a necessary and sufficient condition that I contains such a set A of Lebesgue measure zero that no λ file embraces A is that

$$\mathcal{L}(H) = 0.$$

PROOF. Use Theorems 4 and 5.

7. REMARKS. If λ is such a sequence that

$$\lambda_n = 2^{-1} \cdot 2^{-n} \quad \text{whenever } n \in \omega$$

then

$$\sum_{n \in \omega} \lambda_n = 1$$

and subsum $\lambda = I'$.

Less obvious is the fact that if λ is such a sequence that

$$\lambda_n = 3 \cdot 10^{-1} \cdot (2^{-n} + 4^{-n}) \quad \text{whenever } n \in \omega$$

then

$$\sum_{n \in \omega} \lambda_n = 1$$

and subsum λ is a nondense perfect set of Lebesgue measure $3/5$.

Finally if λ is such a sequence that

$$\lambda_n = 2 \cdot 3^{-1} \cdot 3^{-n} \qquad \text{whenever } n \in \omega$$

then subsum λ is the Cantor ternary set of Lebesgue measure zero.

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BOUNDS FOR THE MODULI OF THE ZEROS OF A POLYNOMIAL

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1. Introduction. Let $f(z) = a_0 + a_1z + \dots + a_nz^n$, $a_0 \neq 0$, $a_n \neq 0$. Following the notation of Ostrowski [1], we define the Newton diagram of $f(z)$ as the broken line D in the (x, y) -plane with the following properties:

- (1) D extends from the point $x=0, y = -\log |a_0|$ to the point $x=n, y = -\log |a_n|$;
- (2) $y_\nu \leq (y_{\nu-1} + y_{\nu+1})/2$ ($\nu=1, 2, \dots, n-1$), where y_ν is the ordinate of D for $x=\nu$;
- (3) The points $(\nu, -\log |a_\nu|)$ ($\nu=0, 1, \dots, n$) all lie on or above D ;
- (4) If (ν, y_ν) is a corner point of D , i.e., $y_\nu < (y_{\nu-1} + y_{\nu+1})/2$, then $y_\nu = -\log |a_\nu|$.

Values of ν for which $(\nu, -\log |a_\nu|)$ is a corner point or an end point of D are called principal indices of $f(z)$. Let $y_\nu = -\log T_\nu$ so $T_\nu = |a_\nu|$ when ν is a principal index, and let $R_\nu = T_{\nu-1}/T_\nu$. The slope of D in the interval $(\nu-1, \nu)$ is $\log R_\nu$.

Let the zeros of $f(z)$ be denoted by z_1, z_2, \dots, z_n with $0 < |z_1| \leq |z_2| \leq \dots \leq |z_n|$, and let b_k^n and B_k^n ($k=1, 2, \dots, n$) be the minimum and maximum respectively of $|z_k|/R_k$ for all polynomials of degree n which do not vanish at the origin. Ostrowski [1] has shown, by considering the polynomial $z^n f(1/z)$, that

$$(1) \qquad B_k^n = 1/b_{n-k+1}^n.$$

Ostrowski proved that b_1^n is the positive root of the equation $1 = x + x^2 + \dots + x^n$, that b_n^n is equal to $1/n$, and that

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