APPLICATIONS OF A THEOREM OF O. SZÁSZ FOR THE PRODUCT OF CESÀRO AND LAPLACE TRANSFORMS

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TO THE LATE PROFESSOR OTTO SZÁSZ

1. **Introduction and notation.** Let $s(u)$ be a function of bounded variation in every finite interval of $u \geqq 0$, assumed (for simplicity) to be such that $s(0)=0$. The Cesàro transform of $s(x)$ of order $\alpha \geqq 0$ is (as usual) defined to be

$$
\begin{aligned}
&C_\alpha(x) \equiv s_\alpha(x) x^{-\alpha} = \alpha x^{-\alpha} \int_0^x (x-u)^{\alpha-1} s(u)\,du, \qquad \alpha > 0, \\
&C_0(x) \equiv s_0(x) = s(x).
\end{aligned} \tag{1}
$$

In a paper [5] some of whose results are generalized here, $C_\alpha(x)$ is called (for convenience) the Riesz integral mean of $s(x)$ of order α.

The Laplace transform of $s(u)$ is defined as

$$
L\{s(u), t\} = \int_0^\infty e^{-tu} d\{s(u)\}, \qquad t > 0, \tag{2}
$$

whenever this integral exists, so that

$$
L\{s(u), t\} = t \int_0^\infty e^{-tu} s(u)\,du, \qquad t > 0, \tag{2'}
$$

the last integral being absolutely convergent [**15**, p. 41, Theorem 2.3a].

Following O. Szász [**13**], we may call

$$
L\{C_\alpha(u), t\} = t \int_0^\infty e^{-tu} C_\alpha(u)\,du, \qquad \alpha > 0,\ t > 0, \tag{3}
$$

which is the Laplace transform of the Cesàro transform of order α of $s(u)$, a product of Cesàro and Laplace transforms of $s(u)$.

The theorem referred to in the title is proved independently in Rajagopal [**5**, Lemma 4] and in Szász [**13**, §3]; it runs as follows.

THEOREM A. *If*

(4) $\qquad L\{s(u), t\}$ *of* (2) *exists and tends to* s *as* $t \to +0$,

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then $L\{C_\alpha(u), t\}$ *of* (3) *exists as an absolutely convergent integral and tends to* s *as* $t \to +0$.

A variant of Theorem A, which in fact includes Theorem A and is implicit in the proof of that theorem, may be stated thus:

THEOREM A′. *If*

$$(5)\qquad \begin{gathered} L_\alpha\{s(u), t\} \equiv \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty e^{-tu} s(u) u^\alpha du, \qquad \alpha > 0, \\ \text{converges absolutely for } t > 0, \\ \lim_{t \to +0} L_\alpha\{s(u), t\} = s, \end{gathered}$$

then $L_0\{s(u), t\}$, *or the integral which represents* $L\{s(u), t\}$ *in* (2′), *is such that*

$$L_0\{s(u), t\} \text{ converges absolutely for } t > 0, \ \lim_{t \to +0} L_0\{s(u), t\} = s.$$

PROOF. A proof can be supplied exactly along the lines of either the proof of Lemma 4 in Rajagopal [**5**, pp. 372–373] or the argument in Szász [**13**, p. 260]. In the former case, the main steps of the proof are the two following. Writing $L_\alpha\{s(u), t\} = L_\alpha(t)$ for the sake of brevity, we have

$$\Gamma(\alpha+1) \int_t^\infty \frac{L_\alpha(x)}{x^{\alpha+1}} (x-t)^{\alpha-1} dx = \Gamma(\alpha) \frac{L_0(t)}{t},$$

$$L_\alpha(t) - L_0(t) = \alpha t \int_t^\infty [L_\alpha(t) - L_\alpha(x)] \frac{(x-t)^{\alpha-1}}{x^{\alpha+1}} dx \to 0 \qquad \text{as } t \to +0,$$

and it follows that $L_0(t) \to s$ since $L_\alpha(t) \to s$. For further details of the proof the reader may consult [**5**], loc. cit.

DEDUCTION OF THEOREM A FROM THEOREM A′. Under condition (4) of Theorem A, Proposition 1 of §2 below enables us to conclude that

$$(6)\qquad \begin{gathered} \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty e^{-tu} s_\alpha(u) du \text{ converges absolutely for } t > 0, \\ \lim_{t \to +0} \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty e^{-tu} s_\alpha(u) du = s. \end{gathered}$$

Hence we can take $s_\alpha(u)$ instead of $s(u)u^\alpha$ in Theorem A′ and thereby establish Theorem A.

It may be pointed out in this connexion that Theorem A is not an isolated result of its kind, Szász himself [**14**] having proved an interesting analogue of the theorem for the product of a regular Hausdorff transform of $\{s_n\}$ and the Borel transform of $\{s_n\}$.

This paper is designed mainly to cover certain aspects of Szász's work on the Laplace transform. One of its results is Theorem II of §4, which is dependent on Theorem A′, and is an "asymptotic" version of Szász's Theorem S of §4, similar to the asymptotic version of Doetsch's Proposition 3 of §2 given by Hardy and Littlewood and quoted as Theorem HL in §4. These asymptotic versions of Szász's and Doetsch's theorems, as well as certain other generalizations of these theorems, may be derived from Theorem II′, another deduction from Theorem A′. The generalizations mentioned last are those stated as Corollaries II′.1, II′.2 in §4; they, together with Theorem I′ of §3, constitute a set of converse theorems for the Cesàro summability of any order $\alpha \geqq 0$ of a function, whose Laplace transform or a modification thereof exhibits a "standard" pattern of behaviour as $t \to +0$. A counterpart of Theorem II′, with a two-sided Tauberian condition replacing the one-sided condition of Theorem II′, is stated as Theorem III′ in §5; it has similar corollaries.

2. **Known auxiliary propositions.** The new theorems of §§3, 4, 5 require for their proofs the following auxiliary propositions.

PROPOSITION 1. *If $L\{s(u), t\}$ as defined in* (2) *exists, then*

$$L\{s(u), t\} = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty e^{-tu} s_\alpha(u) du, \qquad \alpha \geqq 0,\ t > 0,$$

and the integral on the right side is absolutely convergent.

This is a well known result [**15**, pp. 73–74, §§8.1, 8.2].

PROPOSITION 2. *Under condition* (4) *of Theorem* A, *a necessary and sufficient condition for $s(x)$ to converge* (*to s*) *as $x \to \infty$ is*

$$s(x) - C_1(x) \equiv x^{-1} \int_0^x u d\{s(u)\} = o(1), \qquad x \to \infty.$$

This is Schnee's analogue, for the Laplace transform, of a classical theorem of Tauber [**15**, p. 187, Theorem 3b].

PROPOSITION 3. *If $L\{s(u), t\}$ as defined in* (2′) *exists and tends to s as $t \to +0$, and if $s(u) \geqq 0$, then*

$$s_1(x) \sim sx, \quad \text{or} \quad C_1(x) \to s, \qquad \text{as } x \to \infty.$$

This is Doetsch's "positive" Tauberian theorem which may be proved by either Wiener's method [**15**, p. 221, Theorem 14] or Karamata's method [**3**, Theorem 3.82].

3. **Generalization of a classical Tauberian theorem.** The following theorem is due to Karamata [**4**, Satz A and footnote 13 to p. 35].

THEOREM I. *Let $\mathcal{L}(x)$ be a positive and continuous function for $x>0$ such that $\mathcal{L}(ux)\sim\mathcal{L}(x)$, $u>0$, $x\to\infty$. Let $L\{s(u), t\}$ defined as in* (2) *be such that*

$$L\{s(u), t\} \sim st^{-\alpha}\mathcal{L}(t^{-1}), \qquad \alpha \geqq 0,\ t\to +0. \tag{7}$$

Then it follows, from

$$\left.\begin{aligned} &\textit{either} \quad \lim_{1<\lambda\to 1}\ \liminf_{x\to\infty}\ \operatorname*{lower\ bound}_{x\leqq y\leqq\lambda x}\left[\frac{s(y)-s(x)}{x^\alpha\mathcal{L}(x)}\right] \\ &\textit{or} \quad \lim_{1<\lambda\to 1}\ \liminf_{x\to\infty}\ \operatorname*{lower\ bound}_{x\leqq y\leqq\lambda x}\left[\frac{s(y)}{y^\alpha\mathcal{L}(y)}-\frac{s(x)}{x^\alpha\mathcal{L}(x)}\right]\end{aligned}\right\} \geqq 0, \tag{8}$$

that

$$s(x)\sim\frac{s}{\Gamma(\alpha+1)}\,x^\alpha\mathcal{L}(x), \qquad x\to\infty.$$

The following theorem is a consequence of the above.

THEOREM I′. *If, in Theorem* I, (7) *is replaced by*

$$L\{s(u), t\}\sim s\mathcal{L}(t^{-1}), \qquad t\to +0, \tag{9}$$

and (8) *by*

$$\left.\begin{aligned} &\textit{either} \quad \lim_{1<\lambda\to 1}\ \liminf_{x\to\infty}\ \operatorname*{lower\ bound}_{x\leqq y\leqq\lambda x}\left[\frac{y^\alpha C_\alpha(y)-x^\alpha C_\alpha(x)}{x^\alpha\mathcal{L}(x)}\right] \\ &\textit{or} \quad \lim_{1<\lambda\to 1}\ \liminf_{x\to\infty}\ \operatorname*{lower\ bound}_{x\leqq y\leqq\lambda x}\left[\frac{C_\alpha(y)}{\mathcal{L}(y)}-\frac{C_\alpha(x)}{\mathcal{L}(x)}\right]\end{aligned}\right\} \geqq 0,\ \alpha\geqq 0, \tag{10}$$

without any other change, the conclusion will be altered to

$$C_\alpha(x)\sim s\mathcal{L}(x), \qquad x\to\infty.$$

PROOF. By Proposition 1, (9) implies that

$$L\{s_\alpha(u), t\}\sim s\Gamma(\alpha+1)t^{-\alpha}\mathcal{L}(t^{-1}), \qquad t\to +0,$$

which is (7) with $s_\alpha(u)$ in place of $s(u)$ and $s\Gamma(\alpha+1)$ in place of s.

Also (10) is the same as (8) with $s_\alpha(u)$ instead of $s(u)$. Hence, by Theorem I, (9) and (10) lead to the conclusion of Theorem I'.

It will be noticed that, when $\mathcal{L}(x)\equiv 1$ in Theorem I' and the second alternative of hypothesis (10) of the theorem is chosen, we obtain an analogue, for Cesàro and Laplace transforms of a function $s(x)$, of a theorem of Amnon Amir [1, p. 253, Corollary] for Cesàro and Abel transforms of a sequence $\{s_n\}$. When $\mathcal{L}(x)\equiv 1$ and $\alpha=0$, Theorems I and I' both reduce to a classical Tauberian theorem.

The case $\mathcal{L}(x)\equiv 1$ of Theorem I' is in the same class as the special forms assumed by Theorem II' of §4 in its corollaries.

4. Generalizations of a theorem of O. Szász.

Szász's theorem in question [11, Theorem 1], stated below, is supplementary to Propositions 2, 3.

THEOREM S. *Under condition* (4) *of Theorem* A, *and the additional condition*

$$s(x) - C_1(x) = O_L(1), \qquad x \to \infty,$$

we have

$$s_1(x) \sim sx, \quad \text{or} \quad C_1(x) \to s, \qquad \text{as } x \to \infty.$$

The theorem which follows evidently supplements Theorem I in the case $\mathcal{L}(x)\equiv 1$; it reduces to Theorem S when $\alpha=0$ and its corollary (which is obvious enough to need no proof) reduces to Proposition 2 when $\alpha=0$.

THEOREM II. *Let* $L\{s(u), t\}$, *defined as in* (2), *be such that*

$$L\{s(u), t\} \sim st^{-\alpha}, \qquad \alpha \geqq 0,\ t \to +0. \tag{11}$$

Also let

$$\frac{s(x)}{x^\alpha} - \frac{s_1(x)}{x^{\alpha+1}/(\alpha+1))} = O_L(1), \qquad x \to \infty. \tag{12}$$

Then

$$\frac{s_1(x)}{x^{\alpha+1}/(\alpha+1)} \to \frac{s}{\Gamma(\alpha+1)} \qquad \text{as } x \to \infty.$$

In particular, when O_L (1) *in* (12) *is altered to* $o(1)$ *the above conclusion will be simplified to*

$$s(x) \sim \frac{s}{\Gamma(\alpha+1)}\, x^\alpha, \qquad x \to \infty. \tag{13}$$

COROLLARY II. *If* $L\{s(u), t\}$ *as defined in* (2) *exists, then necessary and sufficient conditions for* (13) *to be valid are* (11) *and* (12) *with* $o(1)$ *instead of* $O_L(1)$.

We can deduce Theorem II as well as the immediately following theorem, due to Hardy and Littlewood [4, Satz 1], from Theorem II′ following the latter.

THEOREM HL. *If* $L\{s(u), t\}$ *as defined in* (2) *exists,* $s(u) \geqq 0$, *and*

$$(14) \quad L\{s(u), t\} \equiv t\int_0^\infty e^{-tu}s(u)du \sim st^{-(\alpha-1)} \quad \text{as } t \to +0,\ \alpha > 0,\ s \geqq 0,$$

then

$$s_1(x) \sim \frac{s}{\Gamma(\alpha+1)}x^\alpha, \qquad x \to \infty.$$

THEOREM II′. *Under condition* (5) *of Theorem* A′, i.e. *the condition*

$$(5) \qquad \lim_{t\to+0} \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^\infty e^{-tu}s(u)u^\alpha du = s, \qquad \alpha \geqq 0,$$

where the integral is assumed to converge absolutely for $t>0$, *and the additional condition*

$$(15) \qquad s(x) - \frac{\alpha+1}{x^{\alpha+1}}\int_0^x s(u)u^\alpha du = O_L(1), \qquad x \to \infty,$$

we have

$$C_1(x) \to s \qquad \text{as } x \to \infty,$$

or equivalently

$$\frac{\alpha+1}{x^{\alpha+1}}\int_0^x s(u)u^\alpha du \to s \qquad \text{as } x \to \infty.$$

In the proof of Theorem II′, and in the deduction of Theorems HL and III′ from Theorem II′, we use the following lemmas besides Theorem A′ and the propositions of §2.

LEMMA 1. *If* $\alpha \geqq 0$,

$$(16) \quad f_\alpha(x) \equiv \frac{1}{x}\int_0^x s(u)du - \frac{1}{x}\int_0^x \frac{\alpha+1}{u^{\alpha+1}}du\int_0^u s(v)v^\alpha dv \to s \qquad \text{as } x \to \infty,$$

then $f_0(x) \to (\alpha+1)s$ *as* $x \to \infty$.

PROOF. The case $\alpha=0$ is trivial. In other cases we have, from the definition of $f_\alpha(x)$,

$$x\{f_0(x) - f_\alpha(x)\} = \int_0^x \frac{\alpha+1}{u^{\alpha+1}}\,du \int_0^u s(v)v^\alpha dv - \int_0^x \frac{du}{u}\int_0^u s(v)dv$$

or

$$\frac{d}{dx}\left[x\{f_0(x) - f_\alpha(x)\}\right] = \frac{\alpha+1}{x^{\alpha+1}}\int_0^x s(v)v^\alpha dv - \frac{\alpha+1}{x}\int_0^x s(v)v^\alpha v^{-\alpha}dv$$
$$+ \frac{\alpha}{x}\int_0^x s(v)dv.$$

Integrating by parts the second integral on the right side, we get

$$\frac{d}{dx}\left[x\{f_0(x) - f_\alpha(x)\}\right] = -\frac{\alpha+1}{x}\int_0^x \frac{\alpha}{u^{\alpha+1}}\,du\int_0^u s(v)v^\alpha dv$$
$$+ \frac{\alpha}{x}\int_0^x s(v)dv = \alpha f_\alpha(x).$$

An integration of the above equality from 0 to u gives

$$f_0(u) = f_\alpha(u) + \frac{\alpha}{u}\int_0^u f_\alpha(x)dx$$

whence the required result follows when we let $u\to\infty$.

LEMMA 2. *If* $\alpha\geqq 0$, *then*

$$\lim_{x\to\infty} C_1(x) = s \quad \textit{implies} \quad \lim_{x\to\infty}\frac{\alpha+1}{x^{\alpha+1}}\int_0^x s(u)u^\alpha du = s,$$

and conversely.

PROOF. Ignoring the trivial case $\alpha=0$, we have, by an integration by parts,

$$\frac{\alpha+1}{x^{\alpha+1}}\int_0^x s(u)u^\alpha du = \frac{\alpha+1}{x}\int_0^x s(u)du$$
$$-\frac{\alpha+1}{x^{\alpha+1}}\,\alpha\int_0^x u^{\alpha-1}du\int_0^u s(v)dv$$
$$= (\alpha+1)C_1(x) - \frac{\alpha+1}{x^{\alpha+1}}\,\alpha\int_0^x u^\alpha C_1(u)du$$

whence, letting $x \to \infty$, we establish the first part of the lemma.

To prove the converse part of the lemma we again ignore the trivial case $\alpha = 0$ and get, by an integration by parts,

$$\begin{aligned}(\alpha + 1)C_1(x) &= \frac{\alpha+1}{x}\int_0^x s(u)u^\alpha u^{-\alpha}du \\ &= \frac{\alpha+1}{x^{\alpha+1}}\int_0^x s(u)u^\alpha du + \frac{\alpha}{x}\int_0^x \frac{\alpha+1}{u^{\alpha+1}}\,du\int_0^u s(v)v^\alpha dv\end{aligned}$$

from which we reach the desired conclusion when $x \to \infty$.

LEMMA 3. *If $us(u)$ is nondecreasing for $u \geqq 0$ and if, for $\alpha + 1 > 0$,*

$$\int_0^x s(u)du \sim Ax^{\alpha+1}, \qquad x \to \infty,$$

then

$$s(x) \sim (\alpha + 1)Ax^\alpha, \qquad x \to \infty.$$

This is a well known result [**15**, p. 194, Corollary 4.4b].

The next lemma is a particular case of a classical theorem of M. Riesz [**3**, Theorem 1.71 (C)].

LEMMA 4. *Let $V(u)$ and $W(u)$ be two positive nondecreasing functions of u defined for $u > 0$ and let $s(u)$ be such that*

$$s(u) = O[V(u)], \qquad s_1(u) = o[W(u)], \qquad \textit{as } u \to \infty.$$

Then, for any ϵ such that $0 < \epsilon < 1$,

$$s_\epsilon(u) = o[V^{1-\epsilon}W^\epsilon], \qquad u \to \infty.$$

PROOF OF THEOREM II′. By Theorem A′,

$$\lim_{t\to+0} t\int_0^\infty e^{-tu}s(u)du = s, \tag{17}$$

this integral converging absolutely for $t > 0$. Proposition 1 shows that (17) can be written:

$$\lim_{t\to+0} t^2\int_0^\infty e^{-tu}s_1(u)du \equiv \lim_{t\to+0} t^2\int_0^\infty e^{-tu}C_1(u)udu = s, \tag{18}$$

this integral also converging absolutely for $t > 0$. Applying to (18) Theorem A′ with $s(u) = C_1(u)$, $\alpha = 1$, we get

$$\lim_{t\to+0} t\int_0^\infty e^{-tu}C_1(u)du = s, \qquad \text{or} \qquad \lim_{t\to+0}\int_0^\infty e^{-tu}d\{C_1(u)\} = s, \tag{19}$$

since the integral involved in the last limit exists.[1] Again Proposition 1 gives

$$t\int_0^\infty e^{-tu}s(u)du = t^2\int_0^\infty e^{-tu}s_1(u)du$$

and so, replacing $s(u)$ by $s(u)u^\alpha$, $\alpha>0$, we obtain

$$\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^\infty e^{-tu}s(u)u^\alpha du$$
$$= (\alpha+1)\frac{t^{\alpha+2}}{\Gamma(\alpha+2)}\int_0^\infty e^{-tu}du\int_0^u s(x)x^\alpha dx$$

where, by hypothesis (5), the left-hand member tends to s as $t\to+0$. Hence, from the right-hand member,

$$(\alpha+1)\lim_{t\to+0}\frac{t^{\alpha+2}}{\Gamma(\alpha+2)}\int_0^\infty e^{-tu}g(u)u^{\alpha+1}du = s$$

where

$$g(u) = \frac{1}{u^{\alpha+1}}\int_0^u s(x)x^\alpha dx.$$

The last step yields, when we use Theorem A′ with $(\alpha+1)$ and $g(u)$ in place of α and $s(u)$ respectively,

$$(20)\qquad \begin{aligned}&(\alpha+1)\lim_{t\to+0} t\int_0^\infty e^{-tu}g(u)du\\ &\qquad\equiv (\alpha+1)\lim_{t\to+0} t\int_0^\infty \frac{e^{-tu}}{u^{\alpha+1}}du\int_0^u s(x)x^\alpha dx = s.\end{aligned}$$

From (17) and (20) we see that

$$(21)\qquad \lim_{t\to+0} t\int_0^\infty e^{-tu}\left\{s(u) - \frac{\alpha+1}{u^{\alpha+1}}\int_0^u s(x)x^\alpha dx\right\}du = 0.$$

The multiplier $\{\cdots\}$ of e^{-tu} in (21) is, when $u\to\infty$, $O_L(1)$ by (15). Now Proposition 3 is obviously true with the hypothesis $s(u)=O_L(1)$ instead of $s(u)\geqq 0$; and so we can replace $s(u)$ in Proposition 3 by the multiplier $\{\cdots\}$ of e^{-tu} in (21) and conclude that (16) holds with $s=0$, as also the following (by Lemma 1):

[1] The existence of the second integral in (19) is a consequence of the existence of the integrals in (17) and (18), along with the fact that we may suppose, without loss of generality, $s(u)$ to be zero in some interval $\langle 0, \delta\rangle$, $\delta>0$.

$$f_0(x) \equiv C_1(x) - \frac{1}{x}\int_0^x C_1(u)du \to 0 \qquad \text{as } x \to \infty. \tag{22}$$

From (19) and (22), by an application of Proposition 2 with $s(u)$ replaced by $C_1(u)$, we get at once the conclusion of Theorem II′ in the first form: $C_1(x) \to s$, $x \to \infty$, and therefore (by Lemma 2) in the second form as well.

In the above proof we have supposed that $\alpha > 0$. When $\alpha = 0$, (21) follows directly from (17) and (19) by subtraction, and (22) follows from (21) without an appeal to Lemma 1. Hence the desired conclusion follows as before, Lemma 2 also being superfluous now.

DEDUCTION OF THEOREM II FROM THEOREM II′. In Theorem II′, replace $s(u)$ by $s^*(u)$ defined thus:

$$s^*(u) = \begin{cases} 0 & \text{in some interval } \langle 0, \delta\rangle, \delta > 0, \\ s(u)/u^\alpha & \text{for } u > \delta. \end{cases}$$

The result is Theorem II with $s\Gamma(\alpha+1)$ instead of s. Consequently Theorem II is proved.

DEDUCTION OF THEOREM HL FROM THEOREM II′. First suppose that $\alpha > 1$. (14) can be written:

$$\frac{t^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-tu}s^*(u)u^{\alpha-1}du \sim \frac{s}{\Gamma(\alpha)}, \qquad t \to +0, \tag{14′}$$

where

$$s^*(u) = \begin{cases} 0 & \text{in an interval } \langle 0, \delta\rangle, \delta > 0, \\ s(u)/u^{\alpha-1} & \text{for } u > \delta. \end{cases}$$

From the fact $s^*(u) \geqq 0$ and (14′) it follows that there is a constant $c > 0$ such that, for all small t,

$$\begin{aligned} c \geqq \frac{t^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-tu}s^*(u)u^{\alpha-1}du &\geqq \frac{t^\alpha}{\Gamma(\alpha)}\int_0^{1/t} e^{-tu}s^*(u)u^{\alpha-1}du \\ &\geqq \frac{t^\alpha e^{-1}}{\Gamma(\alpha)}\int_0^{1/t} s^*(u)u^{\alpha-1}du. \end{aligned}$$

Hence, for all large $x = 1/t$,

$$0 < \frac{\alpha}{x^\alpha}\int_0^x s^*(u)u^{\alpha-1}du \leqq ce\Gamma(\alpha+1).$$

The last inequality, in conjunction with $s^*(u) \geqq 0$, gives

$$(23)\qquad s^*(x) - \frac{\alpha}{x^\alpha}\int_0^x s^*(u)u^{\alpha-1}du = O_L(1), \qquad x \to \infty.$$

Now (14′) and (23) are the hypotheses (5), (15) respectively of Theorem II′ with $\alpha+1$, $s(u)$, s replaced by α, $s^*(u)$, $s/\Gamma(\alpha)$ respectively. Hence the conclusion of Theorem II′ in the second form yields

$$\left.\begin{array}{l} \dfrac{\alpha}{x^\alpha}\displaystyle\int_0^x s^*(u)u^{\alpha-1}du \\ \text{or} \\ \dfrac{\alpha}{x^\alpha}\displaystyle\int_0^x s(u)du \equiv \alpha\,\dfrac{s_1(x)}{x^\alpha} \end{array}\right\} \sim \frac{s}{\Gamma(\alpha)}, \qquad x \to \infty,$$

which is the conclusion of Theorem HL.

Next let $0<\alpha\leqq 1$. We get from (14), by an application of Proposition 1,

$$(14'')\qquad \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^\infty e^{-tu}s_1^*(u)u^\alpha du \sim \frac{s}{\Gamma(\alpha+1)}, \qquad t \to +0,$$

where $s_1^*(u)$ is defined in terms of $s_1(u)$ and $\alpha+1$ precisely as $s^*(u)$ in (14′) is defined in terms of $s(u)$ and α. Hence, arguing as before, we get

$$\frac{\alpha+1}{x^{\alpha+1}}\int_0^x s_1(u)du \sim \frac{s}{\Gamma(\alpha+1)}, \qquad x \to \infty,$$

where $us_1(u)$ is obviously nondecreasing. Hence we reach the conclusion of Theorem HL by means of Lemma 3 with $s_1(u)$, $s/\Gamma(\alpha+2)$ instead of $s(u)$, A respectively.

COROLLARY II′.1. *Under the condition*

$$(6)\qquad \lim_{t\to+0}\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^\infty e^{-tu}s_\alpha(u)du = s$$

where the integral is assumed to converge absolutely for $t>0$ and the additional condition

$$(24)\qquad \left.\begin{array}{ll} \textit{either} & C_\alpha(x) - C_{\alpha+1}(x) \\ \textit{or} & C_\alpha(x) \end{array}\right\} = O_L(1), \qquad x \to \infty,$$

we have

$$C_{\alpha+1}(x) \to s \qquad \textit{as } x \to \infty.$$

This corollary with the first alternative of hypothesis (24) can be deduced from Theorem II′ (as Theorem A from Theorem A′) by taking, in Theorem II′, $s_\alpha(u)$ instead of $s(u)\,u^\alpha$ and using the fact:

$$(\alpha + 1) \int_0^x s_\alpha(u)du = s_{\alpha+1}(x).$$

The corollary with the second alternative of (24) follows from the corollary with the first alternative of (24). For, as in the deduction of Theorem HL from Theorem II′, it can be proved that (6) and the second alternative of (24) together imply the first alternative of (24).

COROLLARY II′.2. *Corollary* II′.1 *can be restated with* (6) *alone changed to* (4).

For, (4) implies (6) by Proposition 1.

Corollaries II′.1 and II′.2 furnish generalizations of Theorem S and Proposition 3, alternative to Theorems II and HL. These corollaries, with the first alternative of (24), are directly proved in another paper [**5**] which also gives two of their applications.

Corollary II′.2 with the second alternative of (24) appears elsewhere as a result reached along a different line of argument [**5**, Theorem T]. It has a classical analogue, for Cesàro and Abel transforms of a sequence $\{s_n\}$, a proof of which is given by Amnon Amir [**1**, Theorem 2.5].

5. **Generalization of a theorem supplementary to Szász's.** A theorem supplementary to Theorem S, with $O_L(1)$ in the additional condition of Theorem S replaced by $O(1)$, appears as the case $\alpha=0$ of the next theorem which can be deduced from Theorem II′.

THEOREM III′. *If, in Theorem* II′, *we assume, instead of* (15),

$$s(x) - \frac{\alpha+1}{x^{\alpha+1}} \int_0^x s(u)u^\alpha du = O(1), \qquad x \to \infty, \tag{25}$$

then the conclusion will be changed to

$$C_\epsilon(x) \to s, \qquad \epsilon > 0.$$

PROOF. By Theorem II′, $C_1(x)\to s$ and so, recalling the first theorem of consistency, we note that it suffices to establish the desired conclusion for $0<\epsilon<1$. By (25) and the conclusion of Theorem II′ in the second form, we have $s(u)=O(1)$. Hence we have simultaneously

$$s(u) = O(1), \qquad s_1(u) - su = o(u), \qquad u \to \infty,$$

or, if $S(u)=s(u)-s$, then $S(u)=O(1)$, $S_1(u)=o(u)$, $u\to\infty$. Lemma 4, with $s(u)$, $V(u)$, $W(u)$ replaced by $S(u)$, 1, u respectively, now shows that

$$S_\epsilon(u) = o(u^\epsilon), \quad \text{or} \quad s_\epsilon(u) - su^\epsilon = o(u^\epsilon)$$

which is the conclusion sought for $0<\epsilon<1$.

Theorem III′ has of course the following corollary taking the place of the two corollaries to Theorem II′.

COROLLARY III′.1. *Under the condition either* (6) *or* (4), *and the additional condition*

$$\text{(26)} \qquad \left.\begin{matrix} \textit{either} & C_\alpha(x) - C_{\alpha+1}(x) \\ \textit{or} & C_\alpha(x) \end{matrix}\right\} = O(1)\,, \qquad x\to\infty,$$

we have, for any $\epsilon>0$,

$$C_{\alpha+\epsilon}(x) \to s \qquad \textit{as } x\to\infty.$$

Corollary III′.1, with the choice of the second alternative of its two conditions, has an analogue for Cesàro and Abel transforms of a sequence, proved by Amnon Amir [1, Theorem 3.3].

Applications of the theorems of this paper to Dirichlet's series are exemplified by the following deduction from Corollary III′.1.

COROLLARY III′.2. *Let*

$$0<\lambda_1<\lambda_2<\cdots, \qquad \lambda_n\to\infty,$$

$$s(u) = \begin{cases} a_1+a_2+\cdots+a_n & \textit{for } \lambda_n \leqq u < \lambda_{n+1}, \\ 0 & \textit{for } 0\leqq u<\lambda_1. \end{cases}$$

Let

$$L\{s(u),t\} \equiv \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}, \qquad t>0,$$

tend to s as $t\to+0$. *Then* $\sum a_n$ *is summable*-$R(\lambda_n,\epsilon)$ *to s for any* $\epsilon>0$ *provided* $\lambda>1$ *exists such that*

$$\text{(27)} \qquad \max_{\lambda_n<\lambda_m\leqq\lambda\lambda_n} \left| a_{n+1}+a_{n+2}+\cdots+a_m \right| = O(1) \qquad \textit{as } n\to\infty.$$

To prove Corollary III′.2 we observe that, by well known arguments [2, Lemmas α, β], (27) together with $L\{s(u),t\}=O(1)$, $t\to+0$, implies

$$a_1\lambda_1+a_2\lambda_2+\cdots+a_n\lambda_n = O(\lambda_n), \qquad n\to\infty,$$

or, in terms of $C_\alpha(x)$ which is now Riesz's typical mean of order $\alpha \geqq 0$ of $\{s_n\} \equiv \{s(\lambda_n)\}$ with respect to $\{\lambda_n\}$,

$$C_0(x) - C_1(x) = O(1), \qquad x \to \infty,$$

whence the conclusion of Corollary III′.2 follows from that of Corollary III′.1 in the case $\alpha = 0$.

Repeating in part the arguments which prove Corollary III′.2 and using Theorem S (as in [6]), we can show that *Corollary* III′.2 *has a one-sided analogue in which* (27) *is replaced by the condition that, as* $n \to \infty$,

$$\min_{\lambda_n < \lambda_m \leqq \lambda\lambda_n} (a_{n+1} + a_{n+2} + \cdots + a_m) = O_L(1), \qquad a_n = O_L(1), \tag{28}$$

which includes the special condition, $a_n\lambda_n/(\lambda_n - \lambda_{n-1}) = O_L(1)$, *whose interesting feature is that it ensures also the relation* lim sup $s_n = s$ *as* $n \to \infty$ [**11**, p. 127].

6. **Concluding remarks.** In one section of a publication already referred to [**3**, §3.8], results are proved which are substantially the same as the well known case $\alpha = 0$, $\mathcal{L}(x) \equiv 1$ of Theorem I and the case $\alpha = 0$ of Theorem II and Corollary II, stated here as Theorem S and Proposition 2. The notes on these results [**3**, p. 104] contain a reference to "generalizations of theorems of this section, where $f(\sigma)$ [$= L\{s(u), t\}$ in the present notation with $t = \sigma$] is assumed to tend to infinity like a logarithmico-exponential function" of $1/\sigma$ or $1/t$. It will be observed that, while Theorem I is such a generalization, Theorem II and Corollary II are the only hitherto known generalizations (of the kind in question) of Theorem S and Proposition 2, with logarithmico-exponential function $(1/t)^\alpha$ and not $(1/t)^\alpha \mathcal{L}(1/t)$.

A few references to papers, not already cited, which prove and apply analogues or generalizations of Theorem S for transforms other than the Laplacian, are added here. The analogue of Theorem S for Cesàro and Abel transforms of a sequence and certain applications of the analogue have been given by Szász himself [**10**; **12**]; while the analogues of Theorem S, as well as those of Theorems HL and I, for certain other transforms, and their applications, are to be found in Rajagopal [**7**, **8**; **9**].

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