References

1. E. T. Copson, *Theory of functions of a complex variable*, Oxford, 1935.
2. G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Berlin, Springer, 1937.
3. W. Magnus and F. Oberhettinger, *Special functions of mathematical physics*, New York, Chelsea, 1949.
4. D. V. Widder, *The Laplace transform*, Princeton, 1941.

University of Alberta

ON FUNDAMENTAL MATRIX SOLUTIONS

HYMAN J. ZIMMERBERG

In this note a recent result of Miller and Schiffer [3, §3] will be extended to show how a fundamental matrix solution for the differential equations of a system of two point boundary value problems, written in matrix form as

$$\mathcal{L}[y] \equiv y' - A(x)y = 0, \qquad s[y] \equiv My(a) + Ny(b) = 0, \tag{1}$$

can be represented in terms of the Green's matrix of the system, or in terms of a special generalized Green's matrix in case the system (1) is compatible. For the system (1), $A(x)$ is an $n\times n$ matrix of complex-valued continuous functions of the real variable x on the finite interval $a\leqq x\leqq b$, M and N are $n\times n$ complex constant matrices such that the rank of the $n\times 2n$ matrix $\|M\ N\|$ is n, while the vector y will be treated as an $n\times 1$ matrix.

1. *In this section we shall assume that the system* (1) *is incompatible.* Then a unique Green's matrix exists and is given by (Bliss [1, §5])

$$G(x, t) = \frac{1}{2}\, Y(x)\left[\frac{|x-t|}{x-t}\, I + D^{-1}\Delta\right] Y^{-1}(t), \tag{2}$$

where $Y(x)$ is a fundamental matrix solution of $\mathcal{L}[y]=0$, $D\equiv MY(a)+NY(b)$, $\Delta\equiv MY(a)-NY(b)$, and I is the $n\times n$ identity matrix.

Theorem. *If $G(x, t)$ is the ordinary Green's matrix for an incompatible system* (1) *and $i_1, i_2, \cdots, i_{2n}$ denotes a renumbering of the columns of the $n\times 2n$ matrix $\|M\ N\|$ such that the $i_1, \cdots, i_n$ columns are linearly independent vectors, then the corresponding $i_1, \cdots, i_n$ columns of the $n\times 2n$ matrix $\|G(x, a)\ G(x, b)\|$ may be chosen as the columns for a fundamental matrix solution of the differential equations $\mathcal{L}[y]=0$.*

Presented, in part, to the Society, October 25, 1952, under the title *On the Green's matrix of a differential system*; received by the editors August 7, 1953.

From (2) it is clear that each column of $\|G(x, a)\ G(x, b)\|$ is a solution of $\mathcal{L}[y]=0$, where it is understood that $G(a, a)\equiv G(a^+, a) = \lim_{x\to a^+} G(x, a)$ and $G(b, b)\equiv G(b^-, b)=\lim_{x\to b^-} G(x, b)$. Moreover, n of these columns are linearly independent as one readily verifies directly from (2) that

$$G(x, a)Y(a) - G(x, b)Y(b) \equiv Y(x).$$

The final result on the selection of the n linearly independent columns then follows from the fact that

$$s[\|G(x, a)\ G(x, b)\|] = \|M\ -N\|,$$

which results from the well-known properties of the Green's matrix $MG(a, t)+NG(b, t)=0$, $G(t^+, t)-G(t^-, t)=I$ on $a<t<b$.

2. On the other hand, if the system (1) is *compatible* of index r, $0<r\leqq n$, Reid [**4**, §3] has shown that principal generalized Green's matrices exist for this system. The first systematic study of generalized Green's functions was given by Elliott [2]. Of the family of principal generalized Green's matrices, one may be specialized by choosing particular matrices $\Psi(x)$ and $\Theta(x)$ in the definitions (27) and on page 454 of [4] so that each of the matrices (28) and (39) of [4] is the identity matrix. In view of the simplification induced by such a choice of $\Psi(x)$ and $\Theta(x)$ made below, the associated principal generalized Green's matrix may be termed *the Green's matrix for the compatible system* (1). In this section we shall show how a fundamental matrix solution for $\mathcal{L}[y]=0$ can be represented in terms of this unique Green's matrix for the compatible system (1). *The notation of Reid* [4] *will be employed.*

Let $Y(x)\equiv\|{}^rY_1(x)\ \cdots\ {}^rY_r(x)\ Y_{r+1}(x)\ \cdots\ Y_n(x)\|$ be a fundamental matrix solution of $\mathcal{L}[y]=0$, where the first r columns are r linearly independent vector solutions of the system (1). As $Z(x)\equiv Y^{-1}(x)$ will then be a fundamental matrix solution of the adjoint differential equations (4) of [4], let Γ be an $n\times n$ nonsingular constant matrix such that the first r rows of $\Gamma Z(x)$ will also satisfy the adjoint boundary conditions (5) of [4]. Thus we may choose ${}^rZ(x)$ as

$$(3)\qquad {}^rZ(x) \equiv \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix} \Gamma Z(x),$$

where I_r denotes the $r\times r$ identity matrix. Then the first r columns of $[1/(b-a)]Y(x)\Gamma^{-1}$ may be chosen for the first r columns of $\Psi(x)$, and the first r rows of $[1/(b-a)]Z(x)$ may be chosen for the first r rows of $\Theta(x)$.

Now, if $G(x, t)$ is the given Green's matrix for the compatible system (1) with the above choice of $\Psi(x)$ and $\Theta(x)$, it follows from relation (40) and Theorem 5 of [4] that $K(x, t) \equiv (\partial/\partial t)G(x, t) + G(x, t)A(t)$ on $a \leqq t < x$, $x < t \leqq b$ is equal to $[1/(b-a)]^r Y(x) Z(t)$. In particular, the maximum number of linearly independent columns of $K(x, a)$ on $a \leqq x \leqq b$ may be chosen as a suitable maximal set of r linearly independent vector solutions of the compatible system (1). It is to be noted that when $K(x, t) \equiv 0$, $G(x, t)$ is an ordinary Green's matrix.

In addition, from Theorem 2 of [4] and relation (3) above we have that

$$G(x, a)Y(a) - G(x, b)Y(b) = Y(x) + {}^rY(x)C,$$

where, in Theorem 2, $G_1(x, t)$ is chosen as the function (16) of [4] and $C \equiv U(a)Y(a) - U(b)Y(b)$. Moreover, as $\int_a^b \Theta(x)G(x, t)dx = 0$ from relation (37) of [4],

$$\begin{aligned} 0 &= \int_a^b \Theta(x)[G(x, a)Y(a) - G(x, b)Y(b)]dx \\ &= [1/(b-a)] \int_a^b \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix} Z(x)[Y(x) + {}^rY(x)C]dx \\ &= \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix} + \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix} C; \end{aligned}$$

and thus,

$${}^rY(x)C = {}^rY(x) \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix} C = -{}^rY(x) \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix} = -{}^rY(x).$$

Consequently,

$$G(x, a)Y(a) - G(x, b)Y(b) = \Vert 0 \cdots 0 \quad Y_{r+1}(x) \cdots Y_n(x) \Vert.$$

Form next $W(x, t) \equiv G(x, t) - xN(x, t)$, where $N(x, t) \equiv (\partial/\partial x)G(x, t) - A(x)G(x, t)$ on $a \leqq x < t$, $t < x \leqq b$. As $N(x, t) = -[1/(b-a)] \cdot Y(x)\Gamma^{-1}\,{}^rZ(t)$ from Theorem 3 of [4], one readily verifies that

$$(\partial/\partial x)W(x, t) - A(x)W(x, t) \equiv 0 \text{ on } a \leqq x < t,\ t < x \leqq b,$$

and that

$$W(x, a)Y(a) - W(x, b)Y(b) = \Vert 0 \cdots 0 Y_{r+1}(x) \cdots Y_n(x) \Vert.$$

Thus each column of $\Vert W(x, a)\ W(x, b) \Vert$, where $W(a, a) \equiv W(a^+, a)$ and

$W(b, b) \equiv W(b^-, b)$, is a vector solution of $\mathcal{L}[y]=0$; and, moreover, $n-r$ of these columns are linearly independent solutions of the differential equations of (1) which do not satisfy the boundary conditions $s[y]=0$.

Therefore, if $j_1, \cdots, j_{n-r}$ denotes $n-r$ columns of the $n \times 2n$ matrix $s[\| W(x, a) \; W(x, b)\|]$ which are linearly independent, then the corresponding $j_1, \cdots, j_{n-r}$ columns of $\|W(x, a) \; W(x, b)\|$ together with the r columns obtained above from $K(x, a)$ will form a suitable fundamental matrix solution for $\mathcal{L}[y]=0$.

Bibliography

1. G. A. Bliss, *A boundary value problem for a system of ordinary linear differential equations of the first order*, Trans. Amer. Math. Soc. vol. 28 (1926) pp. 561–584.

2. W. W. Elliott, *Generalized Green's functions for compatible differential systems*, Amer. J. Math. vol. 50 (1928) pp. 243–258.

3. K. S. Miller and M. M. Schiffer, *On the Green's functions of ordinary differential systems*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 433–441.

4. W. T. Reid, *Generalized Green's matrices for compatible systems of differential equations*, Amer. J. Math. vol. 53 (1931) pp. 443–459.

Rutgers University