ON THE INFINITESIMAL GENERATOR OF A SEMIGROUP OF POSITIVE TRANSFORMATIONS WITH LOCAL CHARACTER CONDITION[1]

SOLOMON LEADER

The study of one-dimensional diffusion processes leads to the consideration of the following structure.

Let $\{T_t\}$ $(t>0)$ be a one-parameter semigroup [2] of linear operators on the space C of bounded continuous functions on a closed interval I satisfying the conditions:

(a) If $f(x) \geqq 0$ for all x in I, then $T_t f(x) \geqq 0$ for all x and all t (positivity).

(b) If f vanishes throughout a neighborhood of x, then $T_t f(x) = o(t)$ (local character).

(c) $T_t 1 = 1$ for all t (conservation of mass).

Let Ω be the infinitesimal generator of $\{T_t\}$. That is,

$$\Omega f(x) = \lim_{t \to 0+} \frac{T_t f(x) - f(x)}{t} \tag{1}$$

defined on the set D of those f in C for which the limit exists uniformly in x. We shall also consider the expression (1) wherever the limit exists pointwise.

In this paper the form of Ω at a point x is determined, but the problem of characterizing Ω in the large is left open.

W. Feller [1] has shown that if the functions in D are twice differentiable, then, with the exception of certain singular points, Ω corresponds to a differential operator of the form

$$\Omega f(x) = a(x) \frac{d^2 f(x)}{dx^2} + b(x) \frac{df(x)}{dx}$$

where $a(x) \geqq 0$. However, Feller was dissatisfied with the arbitrary injection of differentiability conditions since the conditions (a), (b), and (c) are independent of the metric on I. He conjectured that, with only the conditions (a), (b), and (c), Ω must be essentially a second order differential operator.

Consider a fixed point x in I. We shall consider only those functions f for which $f(x) = 0$. This is no real restriction since (c) implies

Received by the editors May 6, 1953 and, in revised form, October 22, 1953.

[1] Research sponsored by the Office of Ordnance Research. The author is grateful to Professor William Feller of Princeton University for his assistance and advice.

$\Omega f=\Omega(f-\lambda)$ wherever Ωf exists. The expression "near x" will be used to mean "in some neighborhood of x." The notation $f=o(h)$ will be used whenever, for arbitrary $\epsilon>0$, $|f(z)|\leqq\epsilon|h(z)|$ for all z near x. The functions f_- and f_+, defined by

$$f_-(z) = \begin{cases} f(z) & \text{for} \quad z \leqq x \\ 0 & \text{for} \quad z \geqq x \end{cases} \quad \text{and} \quad f_+ = f - f_-,$$

will also be useful.

We begin with two fundamental lemmas due to Feller [1].

LEMMA A. *If $f\geqq 0$ near x and $\Omega f(x)$ exists, then $\Omega f(x)\geqq 0$. Thus, $\Omega f(z)\geqq 0$ if f has a local minimum at z and $\Omega f(z)\leqq 0$ if f has a local maximum at z.*

PROOF. $|f|-f=0$ near x. Hence $T_t|f|(x)-T_tf(x)=o(t)$ by (b). Since $T_t|f|\geqq 0$ by (a), and $\Omega f(x)=\lim_{t\to 0} T_tf(x)/t=\lim_{t\to 0} T_t|f|(x)/t$, $\Omega f(x)\geqq 0$.

LEMMA B. *If $\Omega v(x)$ exists, $v\geqq 0$ near x, and $f=o(v)$, then $\Omega f(x)$ exists and $\Omega f(x)=0$.*

PROOF. For arbitrary $\epsilon>0$, $-\epsilon v\leqq f\leqq \epsilon v$ near x. From (b), $-\epsilon T_tv(x) \leqq T_tf(x)+o(t)\leqq \epsilon T_tv(x)$. Dividing by t and letting t approach 0,

$$-\epsilon\Omega v(x) \leqq \liminf_{t\to 0} \frac{T_tf(x)}{t} \leqq \limsup_{t\to 0} \frac{T_tf(x)}{t} \leqq \epsilon\Omega v(x).$$

Since ϵ may be arbitrarily small, the conclusion holds.

In order to introduce a metric we prove

THEOREM 1. *If f is in D and $\Omega f(x)\neq 0$, then f is strictly monotone in a right (left) neighborhood of x.*

PROOF. Suppose f is not strictly monotone in any right neighborhood of x. Then, by induction, we can choose a double sequence of points $\{y_n, z_n\}$ such that $y_n\downarrow x$, $z_n\downarrow x$, $f(y_n)=f(z_n)$, and $x\leqq y_{n+1} < z_{n+1} < y_n < z_n$ for all n. Thus, there exist points r_n and s_n in the interior of (y_{n+1}, z_n) such that f has a local minimum at r_n and a local maximum at s_n. By Lemma A, we have $\Omega f(r_n)\geqq 0$ and $\Omega f(s_n)\leqq 0$. Since $r_n\downarrow x$, $s_n\downarrow x$, and since Ωf is continuous for f in D, it follows that $\Omega f(x)=0$, a contradiction. A symmetric argument gives the theorem for a left neighborhood.

An immediate result of Theorem 1 is

COROLLARY 1. *If f is in D and the zeros of f are dense at x, then $\Omega f(x)=0$.*

If there is a function g in D with $\Omega g(x)\neq 0$, then every f in D has right and left derivatives (not necessarily finite) at x with respect to g. The existence of unilateral derivatives is given by

THEOREM 2. *If f and g are in D and $\Omega g(x)\neq 0$, then*

$$\liminf_{z\to x+}\frac{f(z)}{g(z)}=\limsup_{z\to x+}\frac{f(z)}{g(z)}$$

and

$$\liminf_{z\to x-}\frac{f(z)}{g(z)}=\limsup_{z\to x-}\frac{f(z)}{g(z)}\cdot$$

PROOF. Let P denote the set of real numbers ρ for which $\rho g_+\leqq f_+$ near x, and Q denote the set for which $f_+\leqq \rho g_+$ near x. Then $\rho^-=\sup_P \rho$ and $\rho^+=\inf_Q \rho$ are the lower and upper right derivatives of f with respect to g. Suppose $\rho^-\neq\rho^+$. Then consider any finite ρ such that $\rho^-<\rho<\rho^+$ and $\rho\neq\Omega f(x)/\Omega g(x)$. Such a ρ belongs to neither P nor Q. Thus $f_+-\rho g_+(z)<0$ at a set of points dense at x and $f_+-\rho g_+(z)>0$ at a dense set. Hence, the zeros of $f-\rho g$ are dense at x. By Corollary 1, $\Omega f(x)=\rho\Omega g(x)$, a contradiction. Therefore, $\rho^-=\rho^+$. A similar proof holds for the left derivatives.

We now seek to express Ω in terms of such unilateral derivatives.

THEOREM 3. *Let f and g belong to D. If either $f_+=o(g)$ and $f_+\geqq 0$ near x or $f_-=o(g)$ and $f_-\geqq 0$ near x, then $\Omega f(x)\geqq 0$.*

PROOF. If g has its zeros dense at x from the right, then f does likewise for $f_+=o(g)$, so $\Omega f(x)=0$ by Corollary 1. Otherwise we may assume that $g>0$ in a deleted right neighborhood of x. Then for each $\epsilon>0$, $0\leqq f_+\leqq\epsilon g_+$ near x. For z in an arbitrary deleted right neighborhood of x we may set $\epsilon=f(z)/g(z)$. Then $f_+-\epsilon g_+\leqq 0$ near x and $f-\epsilon g(x)=f-\epsilon g(z)=0$, so $f-\epsilon g$ has a local minimum at some point y, $x<y<z$. By Lemma A, $\Omega(f-\epsilon g)(y)\geqq 0$, so $\Omega f(y)\geqq\epsilon\Omega g(y)$. As z converges to x, ϵ converges to 0 and y converges to x giving $\Omega f(x)\geqq 0$. A similar proof gives the theorem with f_+ replaced by f_-.

We can now extend Lemma B to give

THEOREM 4. *Suppose either $g\geqq 0$ or $fg\geqq 0$ near x for f and g in D. Then $f=o(g)$ implies $\Omega f(x)=0$.*

PROOF. Suppose $\Omega f(x)\neq 0$. Then we cannot have $g\geqq 0$ near x, for this would contradict Lemma B. Hence, $fg\geqq 0$ near x with g having neither a maximum nor minimum at x. By Theorem 1, f is strictly monotone near x. Thus, either $f_-\leqq 0$ and $f_+\geqq 0$ or $f_-\geqq 0$ and $f_+\leqq 0$

near x. By Theorem 3, $\Omega f(x)=0$, contradicting our initial assumption.

By means of the preceding theorem, Theorem 2 can be sharpened to give

THEOREM 5. *If u is in D, $\Omega u(x)\neq 0$, and u is monotone near x, then every f in D has finite unilateral derivatives with respect to u.*

PROOF. Suppose f has an infinite right derivative, $\sigma_+=\pm\infty$. Then f must have a finite left derivative, σ_-. Otherwise, $u=o(f)$ and Theorem 4 would give $\Omega u(x)=0$, a contradiction.

Choose σ so that $\sigma_-<\sigma<\infty$ if $\sigma_+=\infty$ and $-\infty<\sigma<\sigma_-$ if $\sigma_+=-\infty$. Let $v=f-\sigma u$. Then, since $u_+=o(v)$, Lemma B gives $\Omega u_+(x)=0$. So $\Omega u(x)=\Omega u_-(x)$. But $\Omega u_-(x)\leqq 0$ if u is increasing, by Lemma A. So $\Omega u(x)<0$. However, $u_+\geqq 0$ near x and $u_+=o(f)$ implying $\Omega u(x)>0$ by Theorem 3, a contradiction.

A similar proof holds for the left derivative.

We can now express Ω at x in terms of an increasing function u and a function v having a minimum at x.

THEOREM 6. *At x one of the following four cases must hold:*

(I) *$\Omega f(x)=0$ for all f in D.*

(II) *For all u in D monotone near x, $\Omega u(x)=0$. There exists v in D with $v\geqq 0$ near x and $\Omega v(x)>0$. Then, for each f in D there exists a number ρ such that*

$$f=\rho v+o(v) \tag{2}$$

and

$$\Omega f(x)=\rho\Omega v(x). \tag{3}$$

(III) *For all v in D with $v\geqq 0$ near x, $\Omega v(x)=0$. There exists u in D with u strictly increasing near x and $\Omega u(x)\neq 0$. Then, for each f in D there exists a number σ such that*

$$f=\sigma u+o(u) \tag{4}$$

and

$$f(x)=\sigma\Omega u(x). \tag{5}$$

(IV) *There exist u and v in D with u strictly increasing near x, $\Omega u(x)\neq 0$, $v\geqq 0$ near x, and $\Omega v(x)>0$ such that each f in D can be expressed in the form*

$$f=\sigma u+\rho v+h \tag{6}$$

where $h=o(u)=o(v)$. Then

(7) $$\Omega f(x) = \sigma\Omega u(x) + \rho\Omega v(x).$$

PROOF. If case (I) does not hold, there exist functions g in D with $\Omega g(x)\neq 0$. Such functions are either strictly monotone near x or have a maximum (or minimum) at x, by Theorem 1. If all such g are of the latter type, we have case (II); if the former type, we have case (III).

In case (II) consider any f in D. If f had no derivative with respect to v, we could choose a finite ρ such that (i) $\rho^- < \rho < \rho^+$ and (ii) $\rho \neq \Omega f(x)/\Omega v(x)$. By (ii), $\Omega(f-\rho v)(x)\neq 0$. So $f-\rho v$ is unilaterally monotone near x, by Theorem 1. By Theorem 2, $f-\rho v$ has unilateral derivatives with respect to v and these derivatives, $\rho^- - \rho$ and $\rho^+ - \rho$, are of opposite sign, by (i). Thus, since $v \geqq 0$ near x, $f-\rho v$ is monotone near x. From the assumptions of case (II), $\Omega(f-\rho v)(x)=0$ which contradicts (ii). Hence, (2) holds. (3) follows from Lemma B.

In case (III) each f has a finite derivative σ with respect to u. Otherwise, there would exist a finite σ such that (i) $\sigma^- < \sigma < \sigma^+$ and (ii) $\sigma \neq \Omega f(x)/\Omega u(x)$. Since f has finite unilateral derivatives σ_-, σ_+ with respect to u by Theorem 5, (i) implies either $f-\sigma u \geqq 0$ or $f-\sigma u \leqq 0$ near x. Hence, (III) gives $\Omega(f-\sigma u)(x)=0$ which contradicts (ii). Thus, we obtain (4) with $\sigma=\sigma_-=\sigma_+$. If $f-\sigma u$ is not monotone near x, then $\Omega(f-\sigma u)(x)=0$ by (III). If $f-\sigma u$ is monotone near x, then $\Omega(f-\sigma u)(x)=0$ by Theorem 4. Hence, (5) holds.

If the unilaterally monotone functions g for which $\Omega g(x)\neq 0$ are of *two* types, some monotone near x and others with extreme values at x, then we may choose a function of each type. In particular, choose u such that u is increasing near x and $\Omega u(x)\neq 0$, and choose v such that $v\geqq 0$ near x and $\Omega v(x)>0$. We then have (IV), which we treat in two cases: one in which v *has* a derivative with respect to u and the other in which v *has no* derivative with respect to u.

Case (IV-1). $v=o(u)$. In this case every f in D has a derivative σ with respect to u. For, suppose $\sigma_-\neq\sigma_+$. Then $s=f-2^{-1}(\sigma_-+\sigma_+)u$ has unilateral derivatives $2^{-1}(\sigma_- - \sigma_+)$ and $2^{-1}(\sigma_+ - \sigma_-)$ which differ in sign. So s has a maximum (or minimum) at x and $v=o(s)$. Lemma B gives $\Omega v(x)=0$, a contradiction.

Let $s=f-\sigma u$, so $s=o(u)$. If s had unequal derivatives $(\rho^- < \rho^+)$ with respect to v, we could choose ρ such that $\rho^- < \rho < \rho^+$ and $\rho \neq \Omega s(x)/\Omega v(x)$. Thus, $\Omega(s-\rho v)(x)\neq 0$. So $s-\rho v$ is unilaterally monotone near x, by Theorem 1. By Theorem 2, $s-\rho v$ has unilateral derivatives with respect to v and these derivatives, $\rho^- - \rho$ and $\rho^+ - \rho$, are of opposite sign. Since $v\geqq 0$ near x, $s-\rho v$ is monotone near x. But $s-\rho v=o(u)$, so $\Omega(s-\rho v)(x)=0$, by Theorem 4, a contradiction. Hence, let $\rho=\rho^-=\rho^+$ which must be finite. For, if $\rho=\infty$ (or $\rho=-\infty$), then

$s \geqq 0$ (or $s \leqq 0$) near x, since $v \geqq 0$ near x, and $v = o(s)$. Theorem 4 would give $\Omega v(x) = 0$, a contradiction. We thus have (6) for case (IV-1).

Case (IV-2). v has no derivative with respect to u ($\beta_- < \beta_+$). We may assume $\beta_+ = 1$ and $\beta_- = -1$. For, given any w in D with $w \geqq 0$ near x, $\Omega w(x) > 0$, and unilateral derivatives ($\alpha_- < \alpha_+$) with respect to u, we can define

$$v = \frac{2}{\alpha_+ - \alpha_-}\left\{w - \frac{1}{2}(\alpha_+ + \alpha_-)u\right\}.$$

Thus, v has $\beta_+ = 1$, $\beta_- = -1$ and $v \geqq 0$ near x. Choose β such that $-1 < \beta < 1$ with sign such that $\beta\Omega u(x) > 0$. Then, $v - \beta u \geqq 0$ near x, so $\Omega(v - \beta u)(x) \geqq 0$ by Lemma A. Thus $\Omega v(x) > 0$.

Now, for f in D with unilateral derivatives σ_-, σ_+ with respect to u, let $\sigma = 2^{-1}(\sigma_+ + \sigma_-)$ and $\rho = 2^{-1}(\sigma_+ - \sigma_-)$. Then it is easily verified by taking right and left derivatives that $f - \sigma u - \rho v = o(u) = o(v)$, thus giving (6). (7) follows from Lemma B.

Thus, Ω will correspond to a second order differential operator at x whenever it is possible to choose u and v such that $u^2 = v + o(v)$.

References

1. W. Feller, *On positivity preserving semigroups of transformations on* $C[r_1, r_2]$, Annales de la Société Polonaise de Mathématique (1952).

2. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, New York, 1948.

Rutgers University