

ON NAKAYAMA'S EXTENSION OF THE $x^{n(x)}$ THEOREMS

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In [6] Nakayama proved that if A is a division ring with center Z such that every element a of A satisfies

$$(1) \quad f(a) = a^{n_1(a)}\alpha_1 + \cdots + a^{n_r(a)}\alpha_r \in Z$$

where the α_i are r fixed nonzero elements of Z , and $0 < n_1(a) < n_2(a) < \cdots < n_r(a)$ ($i=2, \dots, r$), then $A=Z$. In [3, Theorem 11; 5; 2; 1] specialized forms of (1) (e.g. $a^{n(a)} \in Z, a^{n(a)} - a \in Z$) are shown to imply commutativity at least for semi-simple rings. It is natural therefore to seek an extension of Nakayama's result to semi-simple rings. Since a semi-simple ring is a subdirect sum of primitive rings and (1) is preserved under homomorphism we first study primitive rings satisfying (1).

If A is such a ring it may be identified with a dense ring of linear transformations on a vector space \mathfrak{M} over a division ring D . Since D is the ring of endomorphisms of \mathfrak{M} that commute with A , Z is in D and in fact in the center of D . We may assume that there are at least two independent vectors x, y in \mathfrak{M} . If $\lambda \in D$ there is an element a in A such that $xa=0, ya=y\lambda$. Now $ya^n=y\lambda^n$ so that if $f(a) \in Z$ is the relation of type (1) that a satisfies, we have $xf(a)=0, yf(a)=yf(\lambda)$. Since $f(a) \in Z \subset D$ this makes $f(a)=0$ so that $yf(\lambda)=0, f(\lambda)=0$, and by Nakayama's result $D=Z$. Thus the center of a primitive ring satisfying (1) is a field.

Moreover, we have proved that each element in this field satisfies a polynomial equation of the form $f(a)=0$ with f as in (1). Let P be the prime subfield of Z and Q be the field obtained by adjoining to P a maximal, algebraically independent set from among $\alpha_1, \dots, \alpha_r$. Assuming that Z is not absolutely algebraic¹ of prime characteristic, Lemmas 1 and 2 of [6] show that Z is purely inseparable over Q or $Z=Q$. In the former case $f(a)^k=0$ is an equation with coefficients in the rational function field Q ; thus in either case every nonzero rational function λ would satisfy an equation

$$\beta_1 + \beta_2\lambda^{n_2(\lambda)} + \cdots + \beta_r\lambda^{n_r(\lambda)} = 0$$

with $n_i(\lambda) \geq 1$ and the β_i fixed polynomials. [If $Q=P$ = the rationals we take the β_i to be integers.] However the ring of polynomials in several variables [the ring of integers] is a unique factorization do-

Presented to the Society, June 20, 1953; received by the editors May 27, 1953.

¹ Algebraic over its prime subfield.

main and so the numerator of λ would have to divide β_1 for every λ in Q , which is impossible. Hence Z is an absolutely algebraic field of characteristic p .

Thus a primitive ring satisfying (1) is an algebraic algebra over an absolutely algebraic field of characteristic p ; this automatically makes it an algebraic algebra over $GF(p)$. Conversely, every algebraic algebra over $GF(p)$ satisfies a very special form of (1). For if a is an element of A , there is a polynomial f over $GF(p)$ such that $f(a) = 0$. The splitting field of f is a finite field, so that for some $q = p^s$, each linear factor of f divides $x^q - x$. If m is a power of p larger than the multiplicity of each root of f , then f divides $x^{mq} - x^m$. Hence a satisfies $a^{mq} - a^m = 0$. We have

THEOREM 1. *A noncommutative primitive ring satisfies (1) if and only if it is an algebraic algebra over its center² and this center is an absolutely algebraic field of prime characteristic. In this case, the condition (1) implies $a^{n(a)} - a^{m(a)} = 0$, $0 < n(a) < m(a)$.*

Next, if A is a semi-simple ring satisfying (1) with α_1 in no primitive ideal, it is clear that A will be a subdirect sum of rings C and A_p , where C is a commutative semi-simple ring and each A_p is a subdirect sum of noncommutative primitive algebraic algebras over $GF(p)$. However, not every subdirect sum of this last form will satisfy (1): A complete direct sum of an infinite number of copies of the ring of 2×2 matrices over the algebraic closure of $GF(p)$ does not satisfy any condition of type (1). We can also see that (1) cannot always be replaced by $a^{n(a)} - a^{m(a)} = 0$ in such an A_p . To obtain an example, let Z_i be the algebraic closure of $GF(p)$, A_i the ring of all 2×2 matrices over Z_i , and A the local direct sum³ of the A_i with respect to the Z_i . Then for each a in A , $a^{n(a)} - a^{m(a)}$ is in the center of A , but A is clearly not algebraic over any field. In general we do not know whether (1) can always be replaced by the condition $a^{n(a)} - a^{m(a)}$ in the center. However, if the $n_i(a)$ are bounded we can settle the question; in fact:

THEOREM 2. *Let A be a semi-simple ring satisfying (1) with α_1 in no primitive ideal and with bounded $n_i(a)$. Then A is a subdirect sum of a commutative ring and total matrix algebras with a bounded number of*

² We remark that Theorem 1 and its proof remain true verbatim if we replace the center of A throughout by the centroid of A —the ring of endomorphisms of the additive group of A which commute with the left and right multiplications [4, p. 236].

³ Each element of A is an element of the Cartesian product of the A_i having its i th component in Z_i for all but a finite number of i 's.

elements. Furthermore, A satisfies the condition $a^n - a^m \in Z$ with n and m fixed, unequal positive integers.

PROOF. Let k be a bound for $n_i(a)$ and r be as in (1). We show that every noncommutative primitive image B of A has $\leq l$ elements, where l is a function of k and r alone. By our previous arguments each element η of the center Y of B satisfies an equation $f(\eta) = 0$ with f as in (1). Since the degree of f is $\leq k$ there are at most k^r such polynomials. Let g be their product; its degree is at most $s = k^{r+1}$, and every element of Y is a root of g . Hence Y is a field with at most s elements. Now B is a primitive algebraic algebra of degree $\leq k$ over Y and so by [3, Theorems 5 and 8] it is a total matrix algebra of degree $\leq k$ over Y . Thus B has at most $l = s^{k^2}$ elements.

Hence B is an algebraic algebra over $GF(p)$ of dimension $< l$. Therefore every element b of B satisfies a polynomial equation of degree $< l$ and with coefficients in $GF(p)$. The remarks immediately preceding Theorem 1 then show that there exist positive integers u, v depending only on l and p such that $b^u - b^v = 0$ for all b in B . If we write A as a subdirect sum of the rings C and A_p as before, each A_p will satisfy a polynomial identity $a^u - a^v = 0$. Since the noncommutative primitive summands are of bounded size, at most a finite number of distinct A_p 's can occur. To complete the proof it suffices to choose positive, unequal integers m, n such that $x^n - x^m$ is divisible by each of the finite number of polynomials $x^u - x^v$.

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