A VARIATIONAL PROBLEM IN REACTOR THEORY

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1. Introduction. Let C be a bounded measurable set of points in three-dimensional Euclidean space.[1] Let $H(x, y)$ be symmetric in its arguments, in $L_2(C, C)$, and not almost everywhere zero. Let $u(x)$ be a bounded, measurable, non-negative function on C which is not almost everywhere zero. If in addition there exists a non-negative function $v(x)$ in $L_2(C)$ which is not almost everywhere zero such that

$$v(x) = \int_C H(x, y)u(y)v(y)dy, \tag{1}$$

then we shall say that $u(x)$ is in the class U. The problem[2] is to find a function $u_0(x)$ in U which minimizes the integral

$$I[u] = \int_C u(x)dx$$

on U.

If $u_0(x)$ is in U, then it is known [2] that there exists a finite or denumerable sequence λ_n of real characteristic values and a corresponding sequence $v_n(x)$ of characteristic functions in $L_2(C)$ such that

$$\lambda_n \int_C H(x, y)u_0(y)v_n(y)dy = v_n(x),$$

$$\int_C u_0(x)v_m(x)v_n(x)dx = \delta_{mn}.$$

The theorem we wish to prove may be stated as follows:

THEOREM. *If $u_0(x)$ is in U, if $\lambda_n \leqq \lambda_1 < \lambda_0 = 1$ $(n=2, 3, \cdots)$, and if the characteristic solution $v_0(x)$ is constant, then*

$$\int_C u(x)dx \geqq \int_C u_0(x)dx$$

whenever $u(x)$ is in U and $u(x) \geqq \lambda_1 u_0(x)$ with equality holding if and

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[1] The restrictions to three dimensions and to Euclidean space are of course unnecessary.

[2] The physical background of this problem can be found in a paper by Goertzel [1], which also gives examples.

only if

$$\int_C H(x, y)[u(y) - u_0(y)]dy = 0. \tag{2}$$

2. **Proof of the theorem.** Let $u(x)$ be in U and suppose that $v(x)$ is a non-negative function in $L_2(C)$ which is not almost everywhere zero and for which equation (1) holds. By hypothesis

$$\int_C H(x, y)u_0(y)dy = 1. \tag{3}$$

If we multiply equation (1) by $u_0(y)$ and integrate, we find if we make use of the symmetry of H and equation (3) that

$$\int_C [u(x) - u_0(x)]v(x)dx = 0. \tag{4}$$

Define the sequence k_n' so that

$$k_n' = \int_C u(x)v(x)v_n(x)dx.$$

It is clear that $k_0' \geqq 0$. In fact, $k_0' > 0$, since otherwise $u(x)v(x)$ would vanish almost everywhere and so would $v(x)$ by virtue of equation (1).

It follows from equation (4) that

$$\begin{aligned} \int_C [u(x) - u_0(x)]dx = &- \int_C \frac{[u(x) - u_0(x)]v(x)[v(x) - k_0' v_0]}{(k_0' v_0)^2} dx \\ &+ (\lambda_1 - 1) \int_C u_0(x) \left[\frac{v(x) - k_0' v_0}{k_0' v_0}\right]^2 dx \\ &+ \int_C [u(x) - \lambda_1 u_0(x)] \left[\frac{v(x) - k_0' v_0}{k_0' v_0}\right]^2 dx. \end{aligned} \tag{5}$$

It is known [2] that

$$v(x) = \text{l.i.m.} \sum_{n \geqq 0} \frac{k_n' v_n(x)}{\lambda_n},$$

and consequently the first two terms on the right-hand side of equation (5) are

$$\sum_{n \geqq 1} \frac{(1 - \lambda_n) k_n^2}{\lambda_n^2}, \qquad (\lambda_1 - 1) \sum_{n \geqq 1} \frac{k_n^2}{\lambda_n^2},$$

in which $k_n = k_n' / k_0' v_0$. Hence

$$\int_C [u(x) - u_0(x)]dx = \sum_{n \geqq 1} \frac{(\lambda_1 - \lambda_n) k_n^2}{\lambda_n^2} + \int_C [u(x) - \lambda_1 u_0(x)] \left[\frac{v(x) - k_0' v_0}{k_0' v_0}\right]^2 dx \geqq 0, \tag{6}$$

since $\lambda_n \leqq \lambda_1$, $u(x) \geqq \lambda_1 u_0(x)$.

The inequality (6) is strict unless there is a positive integer N such that $\lambda_1 = \lambda_2 = \cdots = \lambda_N$, $k_{N+1} = k_{N+2} = \cdots = 0$, and a measurable subset C_1 of C such that $u(x) = \lambda_1 u_0(x)$ almost everywhere on C_1 while $v(x) = k_0' v_0$ almost everywhere on $C - C_1$. In this case, let $w(x)$ be defined so that $v(x) = k_0' v_0 + w(x)$. Then $w(x) = 0$ almost everywhere on $C - C_1$,

$$\lambda_1 w(x) = \sum_{n=1}^{N} k_n' v_n(x),$$

$$\lambda_1 \int_C H(x, y) u_0(y) w(y) dy = w(x),$$

$$u(x) w(x) = \lambda_1 u_0(x) w(x).$$

Hence we deduce from equations (1) and (3) that

$$\begin{aligned} k_0' v_0 \int_C H(x, y) [u(y) - u_0(y)] dy \\ &= \int_C H(x, y) u(y) [v(y) - w(y)] dy - k_0' v_0 \\ &= v(x) - w(x) - k_0' v_0 = 0, \end{aligned}$$

so that equation (2) must hold when the inequality (6) becomes an equality.

Conversely, if $u(x)$ satisfies (2), we multiply both sides by $u_0(x)$ and integrate. Using equation (3) and the symmetry of H, we deduce that

$$\int_C u(x) dx = \int_C u_0(x) dx.$$

This completes the proof of the theorem.

It should be noted that if $H(x, y)$ is closed in the space of bounded measurable functions, then equation (2) can hold only when $u(x) = u_0(x)$

almost everywhere. This hypothesis is satisfied in all of the physical examples known so far.

In addition it generally happens that $\lambda_1 < 0$, so that the inequality $u(x) \geqq \lambda_1 u_0(x)$ is vacuously true when $u(x)$ is in U, i.e., a true absolute minimum occurs.

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A CHARACTERIZATION OF ANALYTIC FUNCTIONS[1]

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1. Let f be a mapping of an open set D in the xy-plane into the uv-plane where the component functions u and v are continuously differentiable. If the mapping is sense preserving, then the Jacobian, $J(f)$, cannot be negative at any point. If, by analogy with analytic functions, one assumes also that the Jacobian is zero only if the Jacobian matrix has rank zero, then one is led to the study of a family of mappings $\mathfrak{F}$ where

$$(1)\qquad \begin{aligned} f \in \mathfrak{F} &\rightarrow J(f) \geqq 0, \\ J(f) = 0 &\rightarrow \text{rank of } J \text{ is zero.} \end{aligned}$$

The purpose of this paper is in part to show that if any real linear vector space of mappings $\mathfrak{B}$ is contained in $\mathfrak{F}$ and $\mathfrak{B}$ contains a pair of analytic functions whose derivatives are independent on D, then $\mathfrak{B}$ contains only analytic functions.

We first prove an algebraic lemma upon which the whole characterization rests.

2. Let $\mathfrak{C}$ be the vector space of all 2×2 matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

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