THE LIE ALGEBRA OF A SMOOTH MANIFOLD

M. E. SHANKS AND LYLE E. PURSELL

It is well known that certain topological spaces are determined by rings of continuous real functions defined over them [**1**; **2**; **3**],[1] and for differentiable manifolds the functions may be differentiable [**4**; **7**]. In this note we prove that the Lie algebra of all tangent vector fields with compact supports on an infinitely differentiable manifold determines the manifold, and that two such manifolds with isomorphic Lie algebras are differentiably homeomorphic. This Lie algebra has been studied by H. Cartan [**6**], and for the case of analytic manifolds by Chevalley [**5**].

Let X be an infinitely differentiable (smooth) manifold and let R be the real numbers. Denote by D the algebra over R of all infinitely differentiable functions on X with the natural addition and multiplication, and by D_0 the subalgebra of D consisting of those functions with compact supports (that is, which vanish outside of compact sets). A tangent vector field L is a linear operator on D to D which is also an abstract derivative, that is, if a, $b \in R$ and f, $g \in D$, then $L(af+bg)=aL(f)+bL(g)$ and $L(fg)=fL(g)+gL(f)$. The set $\mathcal{L}$ of all tangent vector fields, with the obvious addition and multiplication by real numbers, becomes a Lie algebra[2] over R when the product of L_1 and L_2 is defined by

$$[L_1, L_2](f) = L_1(L_2(f)) - L_2(L_1(f)).$$

If $f^1, \cdots, f^n$ are local coordinates in a neighborhood V of $x \in X$, then there is a neighborhood U of x, contained in V, such that functions $g^1, \cdots, g^n \in D$ exist which are equal to the functions f^i in U. One can then show,[3] for any $L \in \mathcal{L}$ and $f \in D$, that

$$L(f) = \frac{\partial f}{\partial g^i} L(g^i) \tag{1}$$

in the neighborhood U, where the summation convention operates in (1). Hence $L(g^i)$ are the local coordinates of the (contravariant) tangent vector field L. The algebra $\mathcal{L}$ may also be regarded as a

Presented to the Society, September 4, 1952; received by the editors November 28, 1952 and, in revised form, October 25, 1953.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] See for example [**5**] and [**6**].

[3] For the case of analytic functions, a proof is given in [**5**]. For functions in D a proof is given in [**4**].

module over D but this structure for $\mathcal{L}$ is too rich for our purpose as the coefficient ring D itself determines X [7]. We also define the subalgebra $\mathcal{L}_0 \subset \mathcal{L}$ to be those elements of $\mathcal{L}$ which map D into D_0. It is easy to see that $\mathcal{L}_0$ consists of those tangent vector fields with compact supports, that is, all of whose components vanish outside of compact sets.

The reconstruction of X from $\mathcal{L}_0$ follows classical lines, for which information as to the structure of ideals in $\mathcal{L}_0$ is needed.

LEMMA 1. *If $I \neq \mathcal{L}_0$ is an ideal of $\mathcal{L}_0$, then there exists a point $x \in X$ such that all components of every element of I vanish at x.*

PROOF. Suppose that for each x there exists an $L \in I$ such that, in the local coordinates $g^1, \cdots, g^n$, the components $\lambda^i = L(g^i)$ do not all vanish at x. There exist then local coordinates $h^1, \cdots, h^n$ such that the components $\nu^i = L(h^i) \equiv 0$ if $i \neq k$ and $\nu^k \equiv 1$ in a neighborhood of x. To obtain such functions h^i one has only to solve the system of linear ordinary differential equations

$$\lambda^j = \frac{dg^j}{dh^k} \qquad (j = 1, \cdots, n)$$

in a neighborhood U of x and choose additive "constants" of integration $c^j(h^1, \cdots, \tilde{h}^j, \cdots, h^n)$ so that the Jacobian

$$\partial(g^1, \cdots, g^n)/\partial(h^1, \cdots, h^n) \neq 0$$

in U. The functions h^i may then be extended over all of X so as to be infinitely differentiable (at least if the neighborhood U is slightly shrunk). The vector field L in this preferred coordinate system we will call a *local unit vector* in the k-direction.

Let the notation now be chosen so that λ^i are the coordinates of a local unit in the 1-direction. If $M \in \mathcal{L}$ and has components μ^i then the jth component of $[L, M]$ is

$$[L, M]^j = \lambda^i \frac{\partial \mu^j}{\partial g^i} - \mu^i \frac{\partial \lambda^j}{\partial g^i} = \frac{\partial \mu^j}{\partial g^1} \cdot$$

Since we may always find functions μ^j so that $\partial\mu^j/\partial g^1$ are arbitrary near x, it is clear that every element of $\mathcal{L}$ is locally equal to an element of I.

Consider now an arbitrary element $L \in \mathcal{L}_0$ and let S be the support of L. Cover S by a finite set of neighborhoods V_i, $i = 1, \cdots, r$, with compact closures such that $\overline{V}_i \subset U_i$ where U_i is open and $\overline{U}_i$ is compact and with which are associated elements $M_i \in I$ which are local units in V_i with supports contained in $\overline{U}_i$. Let $\{W_i\}$ be an open covering of S with $\overline{W}_i \subset V_i$ with associated functions $f_i \in D$ such that $f_i(x)=1$ for $x \in W_i$ and $f_i(x)=0$ for $x \notin V_i$. Set $L_1=f_1L$ and $L_1^* = L - L_1$. Then there exists $N_1 \in \mathcal{L}_0$ such that $L_1 = [M_1, N_1]$. Now let $L_2 = f_2 L_1^*$ and $L_2^* = L_1^* - L_2 = L - L_1 - L_2$. There exists $N_2 \in \mathcal{L}_0$ such that $L_2 = [M_2, N_2]$. Continuing this process gives at the rth step $L_r^* = 0 = L - L_1 - \cdots - L_r$ or

$$L = \sum_{i=1}^{i=r} [M_i, N_i].$$

From this it follows that $I = \mathcal{L}_0$ and this contradiction completes the proof.

LEMMA 2. *If $I \neq \mathcal{L}_0$ is an ideal and x a point where all components of every element of I vanish and if $L \in I$ and $\lambda^i = L(g^i)$, then all derivatives of the λ^i also vanish at x.*

PROOF. Suppose the contrary. Then for some λ^i and some integer $r>0$, $\partial^r \lambda^i / \partial g^{i_1} \cdots \partial g^{i_r} \neq 0$ at x. Let M be a local unit in the i_r-direction, then the jth component of $[L, M]$ is, near x, equal to

$$[L, M]^j = \lambda^i \frac{\partial \mu^j}{\partial g^i} - \mu^i \frac{\partial \lambda^j}{\partial g^i} = - \frac{\partial \lambda^j}{\partial g^{i_r}} \cdot$$

Therefore $[L, M]$ has at least one component with nonvanishing derivatives of order $r-1$ at x. Choice of successive local units will then give an element of I not all of whose components vanish at x.

The above lemmas yield at once the following theorem.

THEOREM 1. *Each maximal ideal of $\mathcal{L}_0$ consists of all elements of $\mathcal{L}_0$ all of whose components, along with all of their derivatives, vanish at some fixed point $x \in X$.*

Theorem 1 gives a natural one-one correspondence between X and the set X^* of all maximal ideals of $\mathcal{L}_0$. With a suitable topology on X^* this correspondence is a homeomorphism. If $A \subset X$, denote by A^* the corresponding set of maximal ideals. The maximal ideal associated with x will be denoted by I_x. The *Stone topology* on X^* is the closure topology defined by: $\overline{\phi} = \phi$ and if $A^* \neq \phi$ then closure of $A^* = (I \mid I \supset \bigcap_{J \in A^*} J)$. To see that this closure operator defines a topology in X^* which makes the natural mapping a homeomorphism we need but show that $\overline{A}^* =$ closure of A^*. But this last relation follows at once. For if $x \in \overline{A}$, then the components of all elements of the members of $\bigcap_{J \in A^*} J$ vanish at the points of A and hence at x by continuity. Likewise for their derivatives. Thus $I_x \in$ closure of A^* and closure of $A^* \subset \bar{A}^*$. Conversely, if $I_x \notin$ closure of A^* then $I_x \not\supset \bigcap_{J \in A^*} J = K$ and some element of K does not have vanishing components at x. Thus $x \notin \bar{A}$ whence $I_x \notin \bar{A}^*$ and closure of $A^* \supset \bar{A}^*$. This completes the proof of

THEOREM 2. *X^* with the Stone topology is homeomorphic to X.*

THEOREM 3. *Let X and Y be smooth manifolds with isomorphic Lie algebras, then the natural homeomorphism of X onto Y given by Theorem 2 is infinitely differentiable.*

The proof depends on the following lemma.[4]

LEMMA 3. *Let I_0 be the maximal idea of $\mathcal{L}_0$ associated with the point x_0 and L any element of $\mathcal{L}_0$; then L does not vanish at x_0 if and only if*

$$[L, \mathcal{L}_0] + I_0 = \mathcal{L}_0. \tag{2}$$

PROOF. If L does not vanish at x_0, it follows from the proof of Lemma 1 that there exists for every $N \in \mathcal{L}_0$ an $M \in \mathcal{L}_0$ such that $[L, M] = N$ in a neighborhood of x_0. Hence $[L, M] - N \in I_0$. Conversely, if (2) holds and L vanishes at x_0, then for every N, with components ν^i, there is an M, with components μ^i, such that

$$\lambda^i \frac{\partial \mu^j}{\partial g^i} - \mu^i \frac{\partial \lambda^j}{\partial g^i} = \nu^j \tag{3}$$

to within an element of I_0, where λ^i are the components of L, all vanishing at x_0. But this is impossible as may be seen by considering only the terms in (3) which are of the first order in the coordinates.

It is thus possible to characterize vector fields vanishing at a point. Let τ denote the natural homeomorphism of X onto Y given by Theorem 2. Then the isomorphism of the Lie algebras will send a vector field on X vanishing at x_0 into a vector field on Y vanishing at $y_0 = \tau(x_0)$. Since, under τ, functions on Y go into functions on X it suffices to show that this correspondence preserves differentiability. Let $\phi(y)$ be infinitely differentiable, $\psi(x) = \phi(\tau(x))$, and M be a vector field not vanishing at $y_0 = \tau(x_0)$. Let L correspond to M under the isomorphism and consider $\phi M = M_1$ and its image L_1 under the isomorphism. Then $M_1 = \phi(y_0)M + M_0$ where M_0 vanishes at y_0 and hence, by Lemma 3, $L_1 = \psi(x_0)L + L_0$ where L_0 vanishes at x_0. Since L does not vanish at x_0 we may suppose that L is a local unit in the

[4] The authors are much indebted to the referee who suggested the proof of the differentiability and to whom we are indebted for Lemma 3 which is the essential step.

1-direction. Then if λ_1^i are the coordinates of L_1 we have $\lambda_1^1 = f(x_0) \cdot 1$ at x_0. Since the components of L_1 are infinitely differentiable, so is $f(x)$.

References

1. I. Gelfand and A. Kolmogoroff, *On rings of continuous functions on topological spaces*, C. R. (Doklady) Acad. Sci. URSS. vol. 22 (1939) pp. 11–15.

2. M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481.

3. E. Hewitt, *Rings of real-valued continuous functions*. I, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99.

4. L. E. Pursell, *Algebraic structures associated with smooth manifolds*, Thesis, Purdue University, 1952.

5. C. Chevalley, *Theory of Lie groups*, Princeton, 1946.

6. H. Cartan, *Notions d'algèbre differential*, Colloque de Topologie, Bruxelles, 1950, pp. 15–27.

7. M. E. Shanks, *Rings of functions on locally compact spaces*, Bull. Amer. Math. Soc. Abstract 57-4-365.

Purdue University and
The Ohio State University