Corollary. *If* $k=1$ *and* $*e_0xe_\nu^2=x$, *then* $*\boldsymbol{a}xe_0\boldsymbol{b}=*\boldsymbol{a}e_0x\boldsymbol{b}$ *for any* $\boldsymbol{a}$ *and* $\boldsymbol{b}$.

Proof.

$$
\begin{aligned}
*\boldsymbol{a}xe_0\boldsymbol{b} &= *\boldsymbol{a}*e_0e_\nu e_{\nu-1}^2xe_0\boldsymbol{b} \qquad \text{(by (2) with } m=1) \\
&= *\boldsymbol{a}e_0*e_\nu e_{\nu-1}^2xe_0\boldsymbol{b} \\
&= *\boldsymbol{a}e_0x\boldsymbol{b} \qquad \text{(by Theorem Q with } m=1).
\end{aligned}
$$

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TWO THEOREMS ON FINITELY GENERATED GROUPS

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Let G be a group generated by a finite subgroup H and an element b of finite order. If H commutes elementwise with b (for this we shall write $[h, b]=e$ for every $h\in H$ where $[h, b]$ designates $hbh^{-1}b^{-1}$), then clearly G is finite and b is in the center of G.

We consider here the case where, for every $h\in H$, $[[h, b]b]=e$, and prove the following theorem:

Theorem. *Let G be generated by the finite subgroup H and the element b of finite order and, for every $h\in H$, let $[[h, b]b]=e$. Then G is finite and b is in the nil radical of G.*

Proof. For $i=1, 2, \cdots, n$ let h_i be the elements of H. Then $h_i^{-1}bh_i$ are all the conjugates of b; for $bh^{-1}bhb^{-1}=h^{-1}bh$ by virtue of the hypothesis $[[h, b]b]=e$.

It follows from the fact that a finite set of conjugates generate a finite normal subgroup (cf. [1]) that b is contained in a finite normal subgroup K of G. But H is finite and hence so also is G/K; and then finally G is finite.

Furthermore since b is in the center of K, b is in the nil radical of G as was asserted.

We can deduce another result from the fact that $[[g, b]b]=e$ for every $g\in G$ implies that b is in the center of a normal subgroup of G.

Theorem. *Let G be a finitely generated group with the property that if $b_1, \cdots, b_n$ are the generators of G, then $[[g, b_i]b_i]=e$ for every $g\in G$ and for $i=1, 2, \cdots, n$. Then G is nilpotent of class at most n. If furthermore the b_i are of finite order then G is finite.*

Received by the editors September 13, 1953.

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PROOF. For $i=1, 2, \cdots, n$ let B_i be the normal subgroup of G in which b_i is central. Since the B_i are normal subgroups of G, so is each of the intersections $B_{i_1} \cap \cdots \cap B_{i_r}$ normal in G. For $j=1, \cdots, n$ let A_j represent the subgroup of G generated by the product of all possible intersections of j of the B_i at a time; i.e., $A_1 = B_1 B_2 \cdots B_n$, $A_2 = (B_1 \cap B_2)(B_1 \cap B_3) \cdots (B_{n-1} \cap B_n)$, etc., and $A_n = B_1 \cap B_2 \cap \cdots \cap B_n$.

Then A_n is in the center of G; for A_n commutes elementwise with all the generators of G. And for each $r=1, \cdots, n$, A_{r-1}/A_r is in the center of G/A_r. For each component $B_{i_1} \cap \cdots \cap B_{i_{r-1}}$ commutes elementwise with $b_{i_1}, \cdots, b_{i_{r-1}}$ and $(B_{i_1} \cap \cdots \cap B_{i_{r-1}}) \cap B_{i_r} \subset A_r$; hence modulo A_r each component of A_{r-1} is in the center of G/A_r and consequently A_{r-1}/A_r is in the center of G/A_r as asserted. Hence G is nilpotent of class at most n. The finiteness of G follows immediately from this if the b_i are of finite order.

COROLLARY. *If G is a finitely generated group all of whose elements have order* 3, *then G is finite* (*cf.* [2]).

For let a and b be any two elements of G. Then $[[b, a]a] = bab^2aba^2b^2a^2 = (bab)(bab)(bab)(b^2a^2)(b^2a^2)(b^2a^2) = e$.

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