A NOTE ON RINGS WITH CENTRAL NILPOTENT ELEMENTS

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The theorem proved in this note, and its corollary, are designed to improve, and in a sense to bring to a more satisfactory completion, a theorem which we proved in [1].

We prove the

THEOREM. *Let R be a ring such that for every element x in R there exists an integer $n=n(x)$, and a polynomial $p(t)=p_x(t)$ with integer coefficients which depend on x, such that $x^{n+1}p(x)=x^n$. If further all the nilpotent elements of R are in the center of R, then R is commutative.*

PROOF. Since $x^{n+1}p(x)=x^n$, we have that $(x^2p(x)-x)x^{n-1}=0$ (we can assume that $n>1$ for this could always be achieved by multiplying both sides of the equation by x). Now, each term of $(x^2p(x)-x)^{n-1}$ involves x to a power which is at least $n-1$; therefore $(x^2p(x)-x)^n =(x^2p(x)-x)(x^2p(x)-x)^{n-1}=0$. Since $x^2p(x)-x$ is nilpotent, by assumption it must lie in the center of R. This is true for every x in R, so it follows from [2] that R is commutative.

COROLLARY. *Let R be a ring such that every element of R generates a finite subring. If the nilpotent elements of R are all in the center, then R is commutative.*

PROOF. Since x in R generates a finite subring, $x^n=x^m$ for some $n>m$, so the corollary is immediate from the theorem. This corollary is a direct generalization of the second theorem in [1].

REFERENCES

1. I. N. Herstein, *A proof of a conjecture of Vandiver*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 370–371.

2. ———, *The structure of a certain class of rings*, Amer. J. Math. vol. 75 (1953) pp. 864–871.

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