EVERY LINEAR TRANSFORMATION IS A SUM OF NONSINGULAR ONES[1]

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In noncommutative Galois theory, the problem arises[2] as to whether the ring of all linear transformations on a vector space over a division ring is generated by its units (=the nonsingular linear transformations=the automorphisms of the space). Equivalently, is every singular linear transformation a sum of nonsingular ones? The answer is trivially yes when the space is finite-dimensional and the division ring is an infinite (commutative) field, for then it suffices to choose, for each linear transformation α, a nonzero scalar a not a characteristic root of α, and write[3] $\alpha=a\iota+(\alpha-a\iota)$. The proof is not much more difficult in the general finite-dimensional case (cf. Case 1 in the proof of the theorem below). In this note we show that in the absence of restrictions on the vector space, the answer is still yes and in fact that every linear transformation is still a sum of *two* nonsingular ones (with the trivial exception of the identity transformation on a space of two elements).

We use the following notations: "$M\cong_\phi N$" for "ϕ is an isomorphism of the vector space M onto the vector space N." All isomorphisms will be isomorphisms onto. Mappings will be written on the right: $x\alpha$ for the image of x under the mapping α; and hence $\alpha\beta$ for the mapping obtained by applying first α and then β. $M=L\oplus K$ will have its usual significance: M is the sum of its complementary subspaces L and K. If α and β are linear mappings of L and K into a space N, $\alpha\oplus\beta$ will denote the unique linear mapping of M into N which induces α on L and β on K. Note that if α and β are isomorphisms onto complementary subspaces of N, then $\alpha\oplus\beta$ is an isomorphism of M onto N. Notation for direct sum of a collection of subspaces or mappings: $\sum\oplus M_i$, $\sum\oplus\alpha_i$.

LEMMA. *The identity mapping ι on a vector space over a division ring is expressible as the sum of two automorphisms σ and $\iota-\sigma$, unless the vector space is one-dimensional over $GF(2)$.*

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[2] J. Dieudonné, *La théorie de Galois des anneux simples et semi-simples*, Comment. Math. Helv. vol. 21 (1948) pp. 154–184. Cf. p. 169.

[3] We shall use ι throughout for the identity mapping.

Proof. If the space is zero-dimensional, necessarily $\iota=\sigma=0$. If the space is one-dimensional, let σ be represented by the matrix (s) where s is an element of the division ring other than 0 and 1. (The fact that the existence of such an s is necessary, but impossible when the division ring is $GF(2)$, shows that the one-dimensional space over $GF(2)$ really does not satisfy the conclusion of the lemma. It is this fact that prevents our using a proof of the lemma more nearly along the lines sketched in the first paragraph.)

If the space is two-dimensional, σ may be represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

If the space is three-dimensional, σ may be represented by the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

If the space is more than three-dimensional, decompose it into a direct sum of two- and three-dimensional subspaces and use for σ the corresponding direct sum of two- and three-dimensional automorphisms.

Theorem. *Let M and N be vector spaces of equal dimension over a division ring and let α be a linear mapping of M into N. Then α is expressible as a sum $\beta+\gamma$ where β and γ are isomorphisms of M onto N, except when M and N are one-dimensional over $GF(2)$ and α is the lone nonzero linear mapping of M into N.*

Proof. Let K be the kernel of α, L a complement of K in M, R the range of α, and S a complement of R in N. Denote the restriction of α to L by α'. Then α' is an isomorphism of L onto R, so that $\dim L=\dim R$.

Case 1: $\dim K=\dim S$. Then there is an isomorphism ϕ of K onto S. By the lemma, there is an automorphism σ of R such that $\iota-\sigma$ is also an automorphism (unless R is one-dimensional over $GF(2)$—which exception we treat in a moment). Let $\beta=\phi\oplus\alpha'\sigma$ and $\gamma=(-\phi)\oplus\alpha'(\iota-\sigma)$, both defined on $M=K\oplus L$ to $N=S\oplus R$. Then $\beta+\gamma=0\oplus\alpha'=\alpha$, and β and γ are isomorphisms as desired.

Now let the division ring be $GF(2)$ and $\dim R=1$. If $\dim K=0$, we clearly have the exceptional case of the theorem (and, in the proof of the lemma, we have essentially remarked that α is not a sum of isomorphisms). If $\dim K>0$, write $K=K_1\oplus K'$ with $\dim K_1$

$=1$, and, similarly, $S=S_1\oplus S'$ with $\dim S_1=1$ and $K'\cong_{\phi'}S'$. Then α maps $L+K_1$ into $R+S_1$ according to the matrix

$$\begin{pmatrix}1&0\\0&0\end{pmatrix} \text{ which equals } \begin{pmatrix}1&1\\1&0\end{pmatrix}+\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Thus α restricted to $L+K_1$ is the sum of two isomorphisms α_1 and α_2. The required isomorphisms are $\beta=\alpha_1\oplus\phi'$ and $\gamma=\alpha_2\oplus(-\phi')$, each mapping $M=(L+K_1)\oplus K'$ onto $N=(R+S_1)\oplus S'$.

Note that Case 1 always occurs when M is finite-dimensional and in fact whenever α has finite-dimensional range.

Case 2: $\dim K>\dim S$. Write $K=K_1\oplus K_2$ with $K_1\cong_\phi S$. As we just remarked, $\dim R$ is now infinite. Furthermore, $\dim R\geqq\dim K$, else $\dim K>\max(\dim R, \dim S)=\dim N=\dim M$, which is impossible. Since $\dim R\geqq\aleph_0$, we have $\aleph_0 \dim K_2\leqq\aleph_0 \dim R=\dim R$, which allows us to split off ω copies of K_2 from R:

$$R = R_\omega \oplus \sum_{i=0}^{\infty} \oplus R_i \quad\text{with}\quad K_2 \cong_\psi R_0 \quad\text{and}\quad R_i \cong_{\psi_i} R_{i+1}.$$

Moreover, it is no loss of generality to suppose that the dimension of the remainder R_ω is infinite or zero; for the former is automatic unless $\aleph_0 \dim K_2=\dim R$, in which case the sum of the R_i may as well exhaust R, making $R_\omega=0$. Thus the lemma is applicable to R_ω.

Let the automorphism of R_ω given by the lemma be σ and define $\chi=\sigma\oplus\sum\oplus\psi_i$. It is easy to verify that χ and $\iota-\chi$ are isomorphisms of R onto a pair of proper subspaces of R, both of which are complementary to R_0 in R. Then $\beta=\phi\oplus\psi\oplus\alpha'\chi$ and $\gamma=(-\phi)\oplus(-\psi)\oplus\alpha'(\iota-\chi)$ are the required isomorphisms mapping $M=K_1\oplus K_2\oplus L$ onto $N=S\oplus R_0\oplus R\chi$ and onto $N=S\oplus R_0\oplus R(\iota-\chi)$, respectively.

Case 3: $\dim K<\dim S$. Write $S=S_1\oplus S_2$ with $K\cong_\phi S_1$. Necessarily $\dim R$ is infinite and $\dim R=\dim L\geqq\dim S$ (argument as in Case 2). This time we split off $-\omega+\omega$ copies of S_2 from R:

$$R = R_\omega \oplus \sum_{i=-\infty}^{\infty} \oplus R_i \quad\text{with}\quad R_0 \cong_\psi S_2 \quad\text{and}\quad R_i \cong_{\psi_i} R_{i+1}.$$

Define σ on R_ω as in the lemma and construct

$$\chi = \sigma \oplus \left(\sum_{i<0} \oplus \psi_i\right) \oplus \psi \oplus \sum_{i>0} \oplus (\iota - \overset{-1}{\psi_{i-1}}).$$

A straightforward verification shows that χ and $\iota-\chi$ are isomorphisms of R onto $R+S_2$. Then $\beta=\phi\oplus\alpha'\chi$ and $\gamma=(-\phi)\oplus\alpha'(\iota-\chi)$ are the required isomorphisms. This completes the proof.

A. Rosenberg has pointed out to me K. Wolfson's result[4] that the ring of all linear transformations on a vector space of dimension greater than one is generated by its idempotents. This result together with the fact that an idempotent falls in Case 1 and hence is a sum of automorphisms gives a partly independent proof that the ring is generated by its units.

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[4] K. Wolfson, *An ideal-theoretic characterization of the ring of all linear transformations*, Amer. J. Math. vol. 75 (1953) pp. 358–385. Cf. Theorem 5.1.