SOME REMARKS ON ν-TRANSITIVE RINGS AND LINEAR COMPACTNESS

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Johnson in [3] has introduced the concept of a ν-transitive ring which generalizes the notion of a dense ring of linear transformations. We give necessary and sufficient conditions that an abstract ring be isomorphic to a ν-transitive ring which contains finite-valued linear transformations. The condition (2) used here is a modification of one used by Baer [1] in his characterization of the endomorphism ring of a primary Abelian operator group. This condition is also related to the linear compactness of a ring considered as a right module over itself. This enables one to conclude that a primitive ring with minimal ideals which is linearly compact (in any topology in which it is a topological ring) is the ring of all linear transformations of a vector space, and that a primitive Banach algebra is linearly compact only when it is finite dimensional.

A ring $E(F, A)$ of linear transformations of the vector space A over the division ring F is called ν-transitive if to every set of less than $\aleph_\nu$ elements a_j of A, linearly independent over F, and any set of elements b_j of A, in one-one correspondence with the a_j, there exists a transformation σ in $E(F, A)$, such that $a_j\sigma = b_j$ for all j.

Let K be an abstract ring and P an arbitrary subset thereof. The right ideal of all elements k in K which satisfies $Pk=0$ shall be called a right annulet. Now let $W=W(K)$ be the class of all right annulets which are cross-cuts of a finite number of maximal right annulets of K. By a W-coset is meant a coset of an ideal in the set $W(K)$. If $E(F, A)$ is any ring of linear transformations and S is a subspace of A, we denote by $R(S)$ the totality of transformations $\sigma \in E(F, A)$ satisfying $S\sigma = 0$.

THEOREM. *A ring K is isomorphic to a ν-transitive ring containing linear transformations of finite rank if and only if:*

(1) *The socle of K is not a zero ring and is contained in every nonzero two-sided ideal of K.*

$(2)_\nu$ *If Q is any set of W-cosets with the finite intersection property, then any subset of Q containing less than $\aleph_\nu$ elements has a nonvacuous intersection.*

PROOF. Condition (1) is necessary and sufficient that K be isomorphic to a dense ring of linear transformations $E(F, A)$, contain-

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ing linear transformations of finite rank (cf. e.g. [4, Theorem 6.1]). Now assume $(2)_\nu$ and let ϕ be an arbitrary linear transformation of (F, A) and S an arbitrary subspace of A of rank $<\aleph_\nu$. Let $\{a_i\}$ denote a basis for S. Since E is dense, there exist $\phi_i \in E$ such that $\phi_i = \phi$ on the one-dimensional subspace Fa_i. An ideal $J \in W(E)$ if and only if $J = R(T)$ where T is a finite-dimensional subspace of A [2, p. 19]. Thus the cosets $\{R(Fa_i) + \phi_i\}$ are W-cosets and possess the finite-intersection property by density of E. Hence by $(2)_\nu$ there exists σ in E such that $\sigma \in R(Fa_i) + \phi_i$ for all i. That is, $a_i(\sigma - \phi_i) = 0$, $a_i\sigma = a_i\phi$, which is ν-transitivity.

Assume now that $E(F, A)$ is ν-transitive. Let $\{R(S_i) + \sigma_i\}$ be W-cosets with the finite intersection property. It may be assumed without loss of generality that the cardinal number of this set of cosets is less than $\aleph_\nu$. Then the subspace $S = \sum S_i$ has rank less than $\aleph_\nu$ since each S_i is finite-dimensional. By Zorn's Lemma, any set of generators of S contains a basis of S, and we can therefore find a basis $\{u_i\}$ of S such that each u_i is contained in at least one S_i. For each i, select one S_i containing u_i, and find by ν-transitivity $\sigma \in E$ such that $u_i\sigma = u_i\sigma_i$ for each i (where σ_i is the linear transformation associated with the S_i containing u_i). If u_i is also contained in S_j, then $u_i\sigma = u_i\sigma_j$ since the finite intersection property of $\{R(S_i) + \sigma_i\}$ implies $\sigma_i = \sigma_j$ on $S_i \cap S_j$. We must prove $\sigma = \sigma_{i_0}$ on S_{i_0} for any i_0. Let $x \in S_{i_0}$ and write[1]

$$x = \alpha_1 u_{i_1} + \alpha_2 u_{i_2} + \cdots + \alpha_n u_{i_n}, \qquad u_{i_j} \in S_{i_j}.$$

Then $x\sigma = \sum \alpha_j(u_{i_j}\sigma_{i_j})$. By the finite intersection property, we can select τ in $\bigcap_{j=0}^{n}(R(S_{i_j}) + \sigma_{i_j})$. Since $\tau = \sigma_{i_0}$ on S_{i_0}, we have $x\tau = x\sigma_{i_0}$. But $x\tau = \sum \alpha_j(u_{i_j}\tau) = \sum \alpha_j(u_{i_j}\sigma_{i_j}) = x\sigma$. Hence $x\sigma = x\sigma_{i_0}$, so that $\sigma \in \bigcap(R(S_i) + \sigma_i)$, completing the proof.

If a ring $E(F, A)$ is ν-transitive for every ordinal ν it must be the ring $T(F, A)$ of all linear transformations of (F, A). Hence the latter ring may be characterized as a ring satisfying (1) and

(2) Any set of W-cosets with the finite intersection property has a nonvacuous intersection.

The condition (2) is essentially the same as (VII) used by Baer in [1].

If K is a topological ring it shall be called *linearly ν-compact* if $(2)_\nu$ holds where W-cosets are replaced by cosets of closed right ideals. It is *linearly compact* if linearly ν-compact for all ν. (Cf. [5].)

The proof of the theorem implies that a primitive ring which is linearly compact in any topology which makes $R(S)$ (for S a one-

[1] This portion of the proof is due to the referee who kindly pointed out a slight gap in our original version.

dimensional subspace) closed must be the ring of *all* linear transformations of a vector space. This would be true in particular if maximal ideals were closed ($R(S)$ is always maximal), or if the ring contained minimal ideals. For in the latter case $R(S)$ is an annulet [2, p. 19] and thus closed in any topology making the ring a topological ring. Suppose K is a primitive and linearly compact Banach algebra. Then K is continuously isomorphic to a dense and linearly compact algebra of bounded operators in a Banach space. Since $R(S)$ is closed, the algebra of operators contains all linear transformations of the space. But unless the space is finite-dimensional this would include transformations which are not continuous, an obvious contradiction.

The rings $T(F, A)$ which occur as linearly compact primitive rings need not satisfy the minimum condition. For if (F, A) is infinite-dimensional, the ring $T_\nu(F, A)$ (the ring of all linear transformations of (F, A) of rank less than $\aleph_\nu$) is linearly ν-compact in the finite topology. This can be seen as follows: In this topology all closed right ideals have the form $R(T)$, for T a subspace of (F, A) [2, p. 20]. Now assume $\{R(S_i)+\sigma_i\}$ has rank less than $\aleph_\nu$ and possesses the finite intersection property. As in proof of the theorem, there exists $\sigma\in T(F, A)$ such that $\sigma=\sigma_i$ on each S_i, so that $\sigma\in\bigcap(R(S_i)+\sigma_i)$. Although $S=\sum S_i$ is of arbitrary rank r, we have

$$r(S\sigma) = r((\sum S_i)\sigma) \leqq \sum r(S_i\sigma_i) < \aleph_\nu,$$

since $\sigma_i\in T_\nu(F, A)$ and therefore $S_i\sigma_i$ has rank less than $\aleph_\nu$. Put $A=S\oplus Q$ and let σ' be that transformation in $T(F, A)$ which agrees with σ on S and satisfies $Q\sigma'=0$. Then $r(A\sigma')=r(S\sigma)$ so that $\sigma'\in T_\nu(F, A)$, and since it agrees with σ on S we have $\sigma'\in\bigcap(R(S_i)+\sigma_i)$, which completes the proof.

As a special case, we have that $T(F, A)$ is linearly compact in the finite topology. It is of course an easy matter to construct examples of ν-transitive rings which are not linearly ν-compact in the finite topology.

References

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