DEDEKIND SUMS AND LAMBERT SERIES

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1. **Introduction.** Apostol [1] has proved a transformation formula for the function

$$G_p(x) = \sum_{m,n=1}^{\infty} n^{-p} x^{mn} \qquad (|x| < 1),$$

where p is a fixed odd integer >1. In the formula occur the numbers

$$(1.1) \qquad c_s(h, k) = \sum_{\mu \pmod k} \overline{B}_{p+1-s}\left(\frac{\mu}{k}\right)\overline{B}_s\left(\frac{h\mu}{k}\right) \qquad (0 \leqq s \leqq p + 1),$$

where $(h, k) = 1$, the summation is over a complete residue system (mod k), and $\overline{B}_s(x)$ is the Bernoulli function. Put

$$(1.2) \qquad f(h, k; \tau) = \sum_{s=0}^{p+1} \binom{p+1}{s} (k\tau - h)^{p-s} c_s(h, k);$$

then Apostol's formula can be put in the form [2, §2]

$$(1.3) \quad G_p(e^{2\pi i\tau}) = (k\tau - h)^{p-1} G_p(e^{2\pi i\tau'}) + \frac{(2\pi i)^p}{2(p+1)!} f(h, k; \tau),$$

where $\tau = (h'\tau + k')/(k\tau - h)$, $hh' + kk' + 1 = 0$. It is also shown in [2] that (1.3) implies the following transformation formula for $f(h, k; \tau)$:

$$(1.4) \qquad f(h, k; \tau) = \tau^{p-1} f\left(-k, h; -\frac{1}{\tau}\right) + \frac{1}{\tau}(B + \tau B)^{p+1}.$$

The purpose of this note is to give an elementary proof of (1.4) depending on the representation of $c_s(h, k)$ by means of "Eulerian" numbers (see (2.3) below).

2. It is convenient to use, in place of (1.1), the fuller notation

$$(2.1) \qquad c_{r,s}(h, k) = \sum_{\mu \pmod k} \overline{B}_r\left(\frac{\mu}{k}\right)\overline{B}_s\left(\frac{h\mu}{k}\right),$$

where now r, s are arbitrary non-negative integers; thus it is clear that $c_s(h, k) = c_{p+1-s,s}(h, k)$. If we define the Eulerian numbers $H_m(\alpha)$ by means of

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$$(2.2)\qquad \frac{1-\alpha}{e^x-\alpha}=\sum_{m=0}^{\infty}H_m(\alpha)\frac{x^m}{m!},$$

then we have the representation [2, §6]

$$(2.3)\qquad c_{r,s}(h,k)=\frac{B_rB_s}{k^{r+s-1}}+\frac{rs}{k^{r+s-1}}\sum_{\zeta\neq 1}\frac{H_{r-1}(\zeta^h)}{1-\zeta^{-h}}\frac{H_{s-1}(\zeta^{-1})}{1-\zeta},$$

where B_m denotes a Bernoulli number in the even suffix notation and the summation is over all kth roots of unity distinct from 1. The formula (2.3) is proved for $r\geqq 1$, $s\geqq 1$ but if we interpret the sum in the right member as 0 for $rs=0$, then it is easily verified that (2.3) is valid for all non-negative r, s. In the next place we have, using (2.2),

$$\sum_{r,s=0}^{\infty}k^rc_{r,s}(h,k)\frac{x^ry^s}{r!s!}=\frac{x}{e^x-1}\frac{y}{e^{y/k}-1}$$
$$+\,xy\sum_{\zeta\neq 1}\frac{1}{(1-\zeta^{-h})(1-\zeta)}\sum_{r=0}^{\infty}H_r(\zeta^h)\frac{x^r}{r!}\sum_{s=0}^{\infty}H_s(\zeta^{-1})\frac{y^sk^{-s}}{s!}$$
$$=\frac{x}{e^x-1}\frac{y}{e^{y/k}-1}+xy\sum_{\zeta\neq 1}\frac{\zeta^h}{e^x-\zeta^h}\frac{\zeta^{-1}}{e^{y/k}-\zeta^{-1}}$$

and therefore

$$(2.4)\qquad \sum_{r,s=0}^{\infty}k^rc_{r,s}(h,k)\frac{x^r}{r!}\frac{y^s}{s!}=xy\sum_{\zeta}\frac{\zeta^h}{e^x-\zeta^h}\frac{\zeta^{-1}}{e^{y/k}-\zeta^{-1}},$$

where the summation is now over all kth roots of unity.

If we put

$$(2.5)\qquad b_{r,s}(h,k)=\sum_{t=0}^{s}(-1)^{s-t}\binom{s}{t}h^{s-t}c_{r+s-t,t}(h,k)$$

or, what is the same thing,

$$(2.6)\qquad c_{r,s}(h,k)=\sum_{t=0}^{s}\binom{s}{t}h^{s-t}b_{r+s-t,t}(h,k),$$

then a straight-forward computation yields

$$(2.7)\qquad \sum_{r,s=0}^{\infty}k^rc_{r,s}(h,k)\frac{x^r}{r!}\frac{y^s}{s!}=\sum_{r,s=0}^{\infty}b_{r,s}(h,k)\frac{(kx+hy)^r}{r!}\frac{y^s}{s!}.$$

For brevity we put

$$(2.8)\qquad kz=kx+hy$$

and

$$F(h, k; z, y) = \sum_{r,s=0}^{\infty} b_{r,s}(h, k) \frac{z^r}{r!} \frac{y^s}{s!}. \tag{2.9}$$

It follows from (2.4) and (2.7) that

$$F(h, k; kz, y) = xy \sum_{\zeta} \frac{\zeta^h}{e^x - \zeta^h} \frac{1}{e^{y/k}\zeta - 1}. \tag{2.10}$$

3. If η runs through the hth roots of unity, then

$$\frac{h}{x^h - 1} = \sum_{\eta} \frac{\eta}{x - \eta} = \sum_{\eta} \frac{1}{x\eta - 1}. \tag{3.1}$$

Thus

$$\frac{h\zeta^h}{e^x - \zeta^h} = \frac{h}{e^x \zeta^{-h} - 1} = \sum_{\eta} \frac{1}{e^{x/h}\eta\zeta^{-1} - 1} = \sum_{\eta} \frac{\zeta}{e^{x/h}\eta - \zeta}$$

and (2.10) becomes

$$F(h, k; kz, y) = \frac{xy}{h} \sum_{\eta,\zeta} \frac{\zeta}{e^{x/h}\eta - \zeta} \frac{1}{e^{y/k}\zeta - 1}. \tag{3.2}$$

But since

$$\frac{\zeta}{e^{x/h}\eta - \zeta} \frac{1}{e^{y/k}\zeta - 1} = \left(\frac{\eta e^{x/h}}{e^{x/h}\eta - \zeta} + \frac{1}{e^{y/k}\zeta - 1}\right) \frac{1}{e^{x/h+y/k}\eta - 1}$$

the right member of (3.2) is equal to

$$\frac{xy}{h} \sum_{\eta,\zeta} \frac{\eta e^{x/h}}{e^{x/h}\eta - \zeta} \frac{1}{e^{x/h+y/k}\eta - 1} + \frac{xy}{h} \sum_{\eta,\zeta} \frac{1}{e^{y/k}\zeta - 1} \frac{1}{e^{x/h+y/k}\eta - 1}.$$

Then using (2.8) and (3.1) we get

$$\begin{aligned} F(h, k; kz, y) = &- \frac{kxy}{h} \sum_{\eta} \frac{1}{e^{-kx/h}\eta^{-k} - 1} \frac{1}{e^{z/h}\eta - 1} \\ &+ \frac{kxy}{(e^y - 1)(e^z - 1)}. \end{aligned} \tag{3.3}$$

Returning to (2.14), we replace h, k, x, y by k, h, $-kx/h$, z respectively; we get

$$\sum_{r,s=0}^{\infty} c_{r,s}(k, h) \frac{(-kx)^r}{r!} \frac{z^s}{s!} = - \frac{kxz}{h} \sum_{\eta} \frac{1}{e^{-kx/h}\eta^{-k} - 1} \frac{1}{e^{z/h}\eta - 1}. \tag{3.4}$$

But by (2.7) the left-hand member of (3.4) is equal to

$$\sum_{r,s=0}^{\infty} b_{r,s}(k, h) \frac{(kz - kx)^r}{r!} \frac{z^s}{s!} = \sum_{r,s=0}^{\infty} b_{r,s}(k, h) \frac{(hy)^r}{r!} \frac{z^s}{s!}.$$

Comparison with (3.3) leads at once to

$$(3.5)\quad zF(h, k; kz, y) = yF(k, h; hy, z) + (kz - hy) \frac{z}{e^z - 1} \frac{y}{e^y - 1}.$$

4. We now compare coefficients of $z^r y^s$ in both members of (3.5). In view of (2.9) we get at once

$$(4.1)\quad rk^{r-1}b_{r-1,s}(h, k) = sh^{s-1}b_{s-1,r}(k, h) + rkB_{r-1}B_s - shB_rB_{s-1}$$

for all $r, s \geqq 0$. Incidentally this proves Theorem 1 of [2].

If we put

$$(4.2)\quad f_m(h, k; \tau) = \sum_{s=0}^{m} \binom{m}{s} (k\tau - h)^{m-1-s} c_{m-s,s}(h, k),$$

then it follows readily from (2.5) that

$$(4.3)\quad (k\tau - h) f_m(h, k; \tau) = \sum_{s=0}^{m} \binom{m}{s} (k\tau)^{m-s} b_{m-s,s}(h, k).$$

In (4.3) replace h, k, τ by $-k, h, -1/\tau$, respectively; then

$$(4.4)\quad \begin{aligned} &\frac{k\tau - h}{\tau} f_m\left(-k, h; -\frac{1}{\tau}\right) \\ &\qquad = \sum_{s=0}^{m} \binom{m}{s} \left(-\frac{h}{\tau}\right)^{m-s} b_{m-s,s}(-k, h). \end{aligned}$$

But it is clear from (2.1) that

$$c_{r,s}(-k, h) = (-1)^s c_{r,s}(k, h)$$

and therefore (2.5) implies

$$(4.5)\quad b_{r,s}(-k, h) = (-1)^s b_{r,s}(k, h).$$

Hence (4.4) becomes

$$(4.6)\quad \begin{aligned} &\tau^{m-1}(k\tau - h) f_m\left(-k, h; -\frac{1}{\tau}\right) \\ &\qquad = (-1)^m \sum_{s=0}^{m} \binom{m}{s} \tau^s h^{m-s} b_{m-s,s}(k, h). \end{aligned}$$

It follows that

$$
\begin{aligned}
(k\tau - h)&\left\{f_m(h, k; \tau) - (-1)^m\tau^{m-2}f_m\left(-k, h; -\frac{1}{\tau}\right)\right\} \\
&= \sum_{s=0}^{m}\binom{m}{s}\tau^{m-s}\left\{k^{m-s}b_{m-s,s}(h, k) - \frac{s}{m-s+1}h^{s-1}b_{s-1,m-s+1}(k, h)\right\} \\
&= \sum_{s=0}^{m}\binom{m}{s}\tau^{m-s}\left\{kB_{m-s}B_s - \frac{s}{m-s+1}hB_{m-s+1}B_{s-1}\right\} \\
&= k\sum_{s=0}^{m}\binom{m}{s}B_{m-s}B_s - h\sum_{s=1}^{m+1}\binom{m}{s-1}\tau^{m-s}B_{m-s+1}B_{s-1} \\
&= \left(k - \frac{h}{\tau}\right)(B + \tau B)^m,
\end{aligned}
$$

and therefore

$$
(4.7) \qquad f_m(h, k; \tau) = (-1)^m\tau^{m-2}f_m\left(-k, h; -\frac{1}{\tau}\right) + \frac{1}{\tau}(B + \tau B)^m.
$$

In particular for $m = p+1$, p odd, (4.7) reduces to (1.4).

We have therefore proved (1.4) and indeed the more general result (4.7) using only the representation (2.3) and familiar properties of the Bernoulli functions.

REFERENCES

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