

$$\sum_1^{\infty} \frac{2m - \alpha}{\alpha} |k_m^2| \leq 1.$$

For $\alpha = 2n$ this becomes

$$(2.2) \quad \sum_1^{\infty} i |c_{n+i}^2| = |c_{n+1}^2| + 2|c_{n+2}^2| + \dots \\ \leq n + (n-1)|c_1^2| + (n-2)|c_2^2| + \dots + |c_{n-1}^2|.$$

The right side of (2.2) has a finite number of terms, and is therefore finite. By (2.1) the c_i are integers of I . It follows from (2.2) and the restrictions on I that the greatest value of i for which c_i may be different from zero is finite.

LEMMA 2. *All the zeros of the polynomial $h(z)$ lie on the circle $|z| = 1$. The constant term of $h(z)$ is 1 and all its coefficients are integers of I , and therefore either zero or not less than 1 in absolute value. It follows that if any zero of $h(z)$ is different from 1 in absolute value, there is at least one which is less than 1 in absolute value. This is contrary to the hypothesis that $f(z)$, which is equal to $z/[h(z)]^{1/n}$, is regular for $|z| < 1$.*

LEMMA 3. *The degree of $h(z)$ does not exceed $2n$.*

Let

$$h(z) = c_m z^m + c_{m-1} z^{m-1} + \dots + 1, \quad \text{where } |c_m| = 1.$$

Then

$$f'(z) = \frac{(m-n)c_m z^m + (m-n-1)c_{m-1} z^{m-1} + \dots - n}{n[h(z)]^{(n+1)/n}}.$$

If $m > 2n$, then $n/(m-n)$, the absolute value of the product of the finite zeros of $f'(z)$, is less than 1. It follows that $f'(z)$ has at least one zero for $|z| < 1$ and by Bernardi [1], $f(z)$ is not schlicht.

3. The main result.

THEOREM. *The set of functions $f(z)$, defined in §1, consists of z and all the functions $z/[h(z)]^{1/n}$ in which $h(z)$ is a polynomial with coefficients in I satisfying the restrictions of Lemmas 2 and 3.*

It remains to prove the functions $z/[h(z)]^{1/n}$ are schlicht. First, consider the variation of the function

$$f_1(z) = \frac{z}{(z-1)^2}$$

as z goes around the unit circle indented at $z=1$. Put $z=e^{i\theta}$. For $\epsilon < \theta < 2\pi - \epsilon$ then

$$f_1(e^{i\theta}) = \frac{-1}{4 \sin^2 \theta/2}.$$

As θ goes from ϵ to $2\pi - \epsilon$ the argument of $f_1(z)$ remains equal to π . On the indentation we have $z=1 - \epsilon e^{i\phi}$ where $\pi/2 < \phi < 3\pi/2$. But then

$$f_1(z) = \frac{1 - \epsilon e^{i\phi}}{\epsilon^2 e^{2i\phi}}$$

and for sufficiently small ϵ the argument of $f_1(z)$ is -2ϕ . As ϕ goes from $3\pi/2$ to $\pi/2$, the argument of $f_1(z)$ increases by 2π . In short, the argument of $f_1(z)$ along the indented circle is nondecreasing with a total variation of 2π . The same is true for the argument of the function

$$\frac{z}{(z - e^{i\alpha})^2} = e^{-i\alpha} f_1(z e^{-i\alpha}).$$

Suppose now that

$$h(z) = (z - e^{i\alpha_1})^{\beta_1} (z - e^{i\alpha_2})^{\beta_2} \dots (z - e^{i\alpha_k})^{\beta_k}$$

where the $\alpha_1, \alpha_2, \dots, \alpha_k$ are distinct real numbers, and where

$$\sigma_1 = \beta_1 + \beta_2 + \dots + \beta_k \leq 2n.$$

Write

$$\frac{z}{[h(z)]^{1/n}} = z^{1-\sigma/2n} \left[\frac{z}{(z - e^{i\alpha_1})^2} \right]^{\beta_1/2n} \left[\frac{z}{(z - e^{i\alpha_2})^2} \right]^{\beta_2/2n} \dots$$

$$\cdot \left[\frac{z}{(z - e^{i\alpha_k})^2} \right]^{\beta_k/2n}.$$

Since the argument of the left side is the sum of the arguments of the factors on the right side, we see that the argument of the left side is nondecreasing (strictly increasing if $\sigma < 2n$) and that the total variation over the unit circle indented at the points $z=e^{i\alpha_1}, \dots, e^{i\alpha_k}$ is equal to

$$\left(1 - \frac{\sigma}{2n}\right) 2\pi + \frac{1}{2n} (\beta_1 + \dots + \beta_k) 2\pi = 2\pi;$$

consequently, if $\sigma < 2n$, the function is schlicht.

If $\sigma = 2n$, then the unit circle is mapped on the whole plane except

for the straight-line slits going from the points

$$\frac{e^{-i\alpha_j}}{[h(e^{-i\alpha_j})]^{1/n}}, \quad j = 1, 2, \dots, k \text{ to } \infty,$$

and therefore it is also schlicht. This proves the theorem.

4. Additional results.

COROLLARY 1. *If $h(z)$ is a polynomial with complex coefficients, satisfying the restrictions of Lemmas 2 and 3, the function $f(z) = z/[h(z)]^{1/n}$ is schlicht for z within the circle $|z| = 1$.*

This is true since the proof that $f(z)$ is schlicht, given in §3, does not depend on the coefficients of $h(z)$ being integers.

COROLLARY 2. *For $h(z)$, the polynomial of Corollary 1, and a any complex number except zero, the equation $h(z) = z^n/a^n$ has no more than n distinct roots whose absolute values are less than 1.*

Let $\omega_i, i = 1, 2, \dots, n$, be the n th roots of unity, and z_1, z_2, \dots, z_m the roots of the equation $h(z) = z^n/a^n$. For $z = z_i, i = 1, 2, \dots, m$, the corresponding value of $f(z)$ on one of the sheets of its Riemann surface will be one of the $\omega_j a, j = 1, 2, \dots, n$. Were there more than n of the z_i less than 1 in absolute value, then at least one of the $\omega_j a$ would correspond to two distinct values of z within the unit circle and on the same sheet, contrary to the fact that f is schlicht there.

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