LOCAL CONTRACTIONS OF COMPACT METRIC SETS WHICH ARE NOT LOCAL ISOMETRIES

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Following Albert Edrei [1], if X is a compact metric space with metric ρ, f is a mapping of X onto X, and $x \in X$, then x is said to be a *point of contraction* under f relative to X provided that there is a positive number $\mu(x)$ such that if $y \in X$ and $\rho(x, y) < \mu(x)$, then $\rho[f(x), f(y)] \leqq \rho(x, y)$. Further, if each point of X is a point of contraction under f relative to X, f will be said to be a *local contraction* of X. Edrei posed the following question: if X is a compact metric space and f is a contraction of X onto X, is f a local isometry? The purpose of this paper is to answer this question in the negative.

1. **Basic example.** Throughout this section polar coordinates, (r, θ)' are used, W denotes the origin, ϕ denotes the rotation defined by $\phi(r, \theta) = (r, \theta+1)$, and ρ denotes the Euclidean distance function. Let $R = 10/9$, and for each non-negative integer i, let $R_i = \sum_{j=0}^{i} 1/10^j$, and $Q_i = (R_i, 0)$. Let $m_0 = n_0 = 0$ and $P_0 = Q_0$. There exists a positive integer n_1 such that $\rho[\phi^{n_1}(P_0), Q_0] < .1$. Let $m_1 = n_1$, $P'_{m_1} = \phi^{m_1}(P_0)$, and for each integer i, $n_0 < i < n_1$, let $P_i = \phi^i(P_0)$. Let A_1' be the polar angle of P'_{m_1} and let A_1 denote an angle such that $\cos A_1 = (R_0/R_1) \cos A_1'$. Let $P_{m_1} = (R_1, A_1)$.

There exists a positive integer n_2 such that $\rho[\phi^{n_2}(P_{m_1}), Q_1] < .01$. Let $m_2 = m_1 + n_2$, let $P'_{m_2} = \phi^{n_2}(P_{m_1})$, and for each integer i, $n_0 < i < n_2$, let $P_{m_1+i} = \phi^i(P_{m_1})$. Let A_2' be the polar angle of P'_{m_2}, and let A_2 denote an angle such that $\cos A_2 = (R_1/R_2) \cos A_2'$. Let $P_{m_2} = (R_2, A_2)$.

Continuing this process indefinitely we obtain: (1) two sequences $\{n_i\}_{i=1}^{\infty}$, $\{m_i\}_{i=1}^{\infty}$ of positive integers; (2) two sequences $\{P_m\}_{m=0}^{\infty}$, $\{P'_{m_i}\}_{i=1}^{\infty}$ of points; and (3) two sequences $\{A_i'\}_{i+1}^{\infty}$, $\{A_i\}_{i=1}^{\infty}$ of angles. The elements thus obtained have the following properties:

(1) $m_{i+1} = m_i + n_{i+1}$;
(2) $\rho(P'_{m_{i+1}}, Q_i) < 1/10^i$;
(3) $P_{m_i+j} = \phi^j(P_{m_i})$, for $n_0 < j < n_{i+1}$;
(4) $P'_{m_{i+1}} = \phi^{n_{i+1}}(P_{m_i}) = (R_i, A'_{i+1})$;
(5) $P_{m_{i+1}} = (R_{i+1}, A_{i+1})$;
(6) $\cos A_{i+1} = (R_i/R_{i+1}) \cos A'_{i+1}$.

Let C denote the circle, $r = 10/9$, $Q = (10/9, 0)$, $K = \bigcup_{n=0}^{\infty} P_n$, $M = C \cup K$, and $L = \bigcup_{i=1}^{\infty} P_{m_i}$. The sequence $\{P_{m_i}\}$ converges to Q.

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For since $\{Q_i\}$ converges to Q, so does $\{P'_{m_i}\}$. Therefore $\{\cos A'_i\}\to 1$ and so $\{\cos A_i\} = \{(R_{i-1}/R_i)\cos A'_i\}\to 1$.

Let f denote the transformation defined by

$$f(P) = \begin{cases} \phi^{-1}(P), & \text{if } P \in C; \\ P_i, & \text{if for some non-negative } i,\ P = P_{i+1}; \\ P_0, & \text{if } P = P_0. \end{cases}$$

Thus if $P\in M-L$, $f(P)=\phi^{-1}(P)$. Therefore f is a local isometry at all points of $M-Q$. To show that f is a local contraction but not a local isometry, it will suffice to show that for each non-negative integer i, $\rho[f(P_{n_{i+1}}), f(Q)]<\rho(P_{n_{i+1}}, Q)$. But $\rho[f(P_{n_{i+1}}), f(Q)] =\rho[\phi^{-1}(P'_{n_{i+1}}), \phi^{-1}(Q)]=\rho(P'_{n_{i+1}}, Q)<\rho(P_{n_{i+1}}, Q)$, the inequality following from the polar distance formula and relations (4), (5), and (6) above.

2. **Other examples.** In the example above, $f(P_1)=f(P_0)=P_0$ and so f is not a homeomorphism. For each integer $i\leqq 2$ let $B_i=(1/i, 0)$. Let $M'=M\cup W\cup(\bigcup_{n=2}^{\infty}B_n)$, let $f'=f$ on $M-P_0$, let $f'(P_0)=B_2$, let $f'(B_i)=B_{i+1}$, and let $f'(W)=W$. Then M' is a compact set in the plane and f' is a homeomorphism of M' onto M' which is a local contraction but not a local isometry.

Using cylindrical coordinates, (r, θ, z), consider the plane of the basic example to be the graph of $z=1$, and consider ϕ to be the rotation defined by $\phi[(r, \theta, z)]\ (r, \theta+1, z)$. Let M' be the cone over M with vertex $(0, 0, 0)$, i.e., $M'=\{(rz, \theta, z)\mid z\in[0, 1]$ and $(r, \theta, 1)\in M\}$. Let f' denote the linear extension of f, i.e., $f'[(zr, \theta, z)]=(zr', \theta', z)$ where $f(r, \theta, 1)=(r', \theta', 1)$. To show that f' is a local contraction of M' onto M', let $L_n=\{(zr, \theta, z)\mid z\in[0, 1]$ and $(r, \theta, 1)=P_n\}$. Then for each integer $i\geqq 0$, L_i is a line interval intersecting the closure of $M'-L_i$ only in the point $(0, 0, 0)$, $f'=\phi$ on $M'-\bigcup_{i=0}^{\infty}L_{m_i}$, and each point of L_0 is fixed. As, for each $i\geqq 0$, f' is linear on $L_{m_{i+1}}$ and as $L_{m_{i+1}}$ and $f'(L_{m_{i+1}})=L_{m_{i+1}-1}$ have lengths $(1+R_{i+1}^2)^{1/2}$ and $(1+R_i^2)^{1/2}$ respectively, each point of $L_{m_{i+1}}$ is a point of contraction under f'. As $\lim_{i\to\infty}L_{m_i}=L_Q=\{(zR, 0, z)\mid z\in[0, 1]\}$, it will suffice to show that if $P\in L_{m_{i+1}}-(0, 0, 0)$ and $P'\in L_Q$, then $\rho[f'(P), f'(P')]<\rho(P, P')$. Then let $P=(zR_{i+1}, A_{i+1}, z)$, $z\neq 0$, and $P'=(yR, 0, y)$; $f'(P) =(zR_i, A'_{i+1}, z)$ and $f'(P')=(yR, -1, y)$. Therefore $\rho[f'(P), f'(P')] =\rho[(zR_i, A'_{i+1}-1, z), (yR, -1, y)]=\rho[(zR_i, A'_{i+1}, z), (yR, 0, y)] <\rho(P, P')$, the inequality following from the cylindrical distance formula and the relations (4), (5), (6) above and $z\neq 0$. Thus f' is a local contraction of a compact continuum onto itself which is not a local isometry.

There exists a totally disconnected compact metric set C and an isometry f of C onto C such that for each point $x \in C$, the iterations $f(x), f^2(x), f^3(x), \cdots$, are dense in C. (E.g., see Vietoris [2], and let the metric ρ be defined, using the triadic system, as follows:

$$\begin{aligned}
&\text{for } 0 \leqq x \leqq 0.1, && 0.2 \leqq y \leqq 1, && \rho(x, y) = 1/2;\\
&\text{for } 0 \leqq x \leqq 0.01, && 0.02 \leqq y \leqq 0.1, && \rho(x, y) = 1/4;\\
&\text{for } 0.2 \leqq x \leqq 0.21, && 0.22 \leqq y \leqq 1, && \rho(x, y) = 1/4;\\
&\text{for } 0 \leqq x \leqq 0.001, && 0.002 \leqq y \leqq 0.01, && \rho(x, y) = 1/8;
\end{aligned}$$

$\cdots$.)

There exists a countable sequence of points $\{P_i\}$ in $C \times [0, 1]$ which converges to a subset of C_0 (the "base set" in $C \times [0, 1]$) in a manner completely analogous to the convergence of $\{P_n\}$ in the basic example. Let $M' = C_0 \cup (\bigcup_{i=0}^{\infty} P_i)$ and let f' be defined in a manner similar to f in the basic example. Then f' is a local contraction of a totally disconnected compact metric set onto itself which is not a local isometry. M' and f' in these last two examples can be extended as above to yield homeomorphisms.

Bibliography

1. Albert Edrei, *On mappings which do not increase small distances*, Proc. London Math. Soc. (3) vol. 2 (1952) p. 272.

2. L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildung*, Math. Ann. vol. 97 (1927) p. 459.

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