LOCAL CONTRACTIONS OF COMPACT METRIC SETS WHICH ARE NOT LOCAL ISOMETRIES

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Following Albert Edrei [1], if X is a compact metric space with metric ρ, f is a mapping of X onto X, and x ∈ X, then x is said to be a point of contraction under f relative to X provided that there is a positive number μ(x) such that if y ∈ X and ρ(x, y) < μ(x), then ρ[f(x), f(y)] ≤ ρ(x, y). Further, if each point of X is a point of contraction under f relative to X, f will be said to be a local contraction of X. Edrei posed the following question: if X is a compact metric space and f is a contraction of X onto X, is f a local isometry? The purpose of this paper is to answer this question in the negative.

1. Basic example. Throughout this section polar coordinates, (r, θ)', are used, W denotes the origin, φ denotes the rotation defined by φ(r, θ) = (r, θ + 1), and ρ denotes the Euclidean distance function. Let R = 10/9, and for each non-negative integer i, let R_i = \sum_{j=0}^{i} 1/10^j, and Q_i = (R_i, 0). Let m_0 = n_0 = 0 and P_0 = Q_0. There exists a positive integer n_1 such that ρ[φ^n_1(P_0), Q_0] < .1. Let m_1 = n_1, P'_m = φ^n_1(P_0), and for each integer i, n_0 < i < n_1, let P_i = φ^n_i(P_0). Let \( A'_i \) be the polar angle of \( P'_m \) and let \( A'_1 \) denote an angle such that \( \cos A'_1 = (R_0/R_1) \cos A_1' \). Let \( P'_m = (R_1, A_1) \).

There exists a positive integer n_2 such that ρ[φ^n_2(P'_m), Q_1] < .01. Let m_2 = m_1 + n_2, let P'_m = φ^n_2(P'_m), and for each integer i, n_0 < i < n_2, let P'_m_i = φ^n_i(P'_m). Let \( A'_2 \) be the polar angle of \( P'_m \), and let \( A'_2 \) denote an angle such that \( \cos A'_2 = (R_1/R_2) \cos A'_2 \). Let \( P'_m = (R_2, A_2) \).

Continuing this process indefinitely we obtain: (1) two sequences \{n_i\}_{i=1}^{∞}, \{m_i\}_{i=1}^{∞} of positive integers; (2) two sequences \{P'_m\}_{m=0}^{∞}, \{P'_m\}_{m=1}^{∞} of points; and (3) two sequences \{A'_i\}_{i=1}^{∞}, \{A'_i\}_{i=1}^{∞} of angles. The elements thus obtained have the following properties:

1. \( m_{i+1} = m_i + n_{i+1} \);
2. \( ρ(P'_m, Q_i) < 1/10^i \);
3. \( P'_m_j = φ^j(P'_m) \), for \( n_0 < j < n_{i+1} \);
4. \( P'_m = φ^{n_{i+1}}(P'_m) = (R_i, A'_i) \);
5. \( P'_m_i = (R_{i+1}, A'_i) \);
6. \( \cos A = (R_i/R_{i+1}) \cos A'_i \).

Let \( C \) denote the circle, \( r = 10/9, Q = (10/9, 0), K = \cup_{n=0}^{∞} P_n, M = C \cup K, \) and \( L = \cup_{m=1}^{∞} P_m \). The sequence \{P'_m\} converges to \( Q \).

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For since \( \{Q_i\} \) converges to \( Q \), so does \( \{P'_m\} \). Therefore \( \{\cos A'_i\} \to 1 \) and so \( \{\cos A_i\} = \{\frac{(R_{i-1}/R_i) \cos A'_i}{\} \to 1 \} \).

Let \( f \) denote the transformation defined by

\[
f(P) = \begin{cases} 
\phi^{-1}(P), & \text{if } P \in C; \\
P_i, & \text{if for some non-negative } i, P = P_{i+1}; \\
P_0, & \text{if } P = P_0.
\end{cases}
\]

Thus if \( P \in M - L \), \( f(P) = \phi^{-1}(P) \). Therefore \( f \) is a local isometry at all points of \( M - Q \). To show that \( f \) is a local contraction but not a local isometry, it will suffice to show that for each non-negative integer \( i \), \( \rho[f(P_{ni+1}), f(Q)] \leq \rho(P_{ni+1}, Q) \). But \( \rho[f(P_{ni+1}), f(Q)] = \rho[\phi^{-1}(P_{ni+1}), \phi^{-1}(Q)] = \rho(P_{ni+1}, Q) \leq \rho(P_{ni+1}, Q) \), the inequality following from the polar distance formula and relations (4), (5), and (6) above.

**2. Other examples.** In the example above, \( f(P_1) = f(P_0) = P_0 \) and so \( f \) is not a homeomorphism. For each integer \( i \geq 2 \) let \( B_i = (1/i, 0) \). Let \( M' = M \cup M \cup (\bigcup_{n=2}^\infty B_n) \), let \( f' = f \) on \( M - P_0 \), let \( f'(P_0) = B_2 \), let \( f'(B_i) = B_{i+1} \), and let \( f'(W) = W \). Then \( M' \) is a compact set in the plane and \( f' \) is a homeomorphism of \( M' \) onto \( M' \) which is a local contraction but not a local isometry.

Using cylindrical coordinates, \((r, \theta, z)\), consider the plane of the basic example to be the graph of \( z = 1 \), and consider \( \phi \) to be the rotation defined by \( \phi[(r, \theta, z)] = (r, \theta + 1, z) \). Let \( M' \) be the cone over \( M \) with vertex \((0, 0, 0)\), i.e., \( M' = \{(rz, \theta, z) | z \in [0, 1] \} \) and \((r, \theta, 1) \in M' \).

Let \( f' \) denote the linear extension of \( f \), i.e., \( f'[(rz, \theta, z)] = (zr', \theta', z) \) where \( f(r, \theta, 1) = (r', \theta', 1) \). To show that \( f' \) is a local contraction of \( M' \) onto \( M' \), let \( L_n = \{(rz, \theta, z) | z \in [0, 1] \} \) and \((r, \theta, 1) = P_n \). Then for each integer \( i \geq 0 \), \( L_i \) is a line interval intersecting the closure of \( M' - L_i \) only in the point \((0, 0, 0)\), \( f' = \phi \) on \( M' - \bigcup_{i=0}^\infty L_{mi} \), and each point of \( L_0 \) is fixed. As, for each \( i \geq 0, f' \) is linear on \( L_{mi+1} \) and as \( L_{mi+1} \) and \( f'(L_{mi+1}) = L_{mi+1} \) have lengths \((1 + R_{i+1}^2)^{1/2} \) and \((1 + R_i^2)^{1/2} \) respectively, each point of \( L_{mi+1} \) is a point of contraction under \( f' \). As \( \lim_{i \to \infty} L_m = L_0 = \{(rz, 0, z) | z \in [0, 1] \} \), it will suffice to show that if \( P \in L_{mi+1} \) and \( P' \in L_0 \), then \( \rho[f'(P), f'(P')] < \rho(P, P') \). Then let \( P = (rz_{i+1}, A_{i+1}, z) \), \( z \neq 0 \), and \( P' = (yR, 0, y) \); \( f'(P) = (rz_{i+1}, A_{i+1}, z) \) and \( f'(P') = (yR, -1, y) \). Therefore \( \rho[f'(P), f'(P')] = \rho[(rz_{i+1}, A_{i+1}, z), (yR, -1, y)] = \rho[(rz_{i+1}, A_{i+1}, z), (yR, 0, y)] < \rho(P, P') \), the inequality following from the cylindrical distance formula and the relations (4), (5), (6) above and \( z \neq 0 \). Thus \( f' \) is a local contraction of a compact continuum onto itself which is not a local isometry.
There exists a totally disconnected compact metric set $C$ and an isometry $f$ of $C$ onto $C$ such that for each point $x \in C$, the iterations $f(x), f^2(x), f^3(x), \ldots$, are dense in $C$. (E.g., see Vietoris [2], and let the metric $\rho$ be defined, using the triadic system, as follows:

\[
\begin{align*}
\text{for } 0 \leq x \leq 0.1, & \quad 0.2 \leq y \leq 1, & \quad \rho(x, y) = 1/2; \\
\text{for } 0 \leq x \leq 0.01, & \quad 0.02 \leq y \leq 0.1, & \quad \rho(x, y) = 1/4; \\
\text{for } 0.2 \leq x \leq 0.21, & \quad 0.22 \leq y \leq 1, & \quad \rho(x, y) = 1/4; \\
\text{for } 0 \leq x \leq 0.001, & \quad 0.002 \leq y \leq 0.01, & \quad \rho(x, y) = 1/8; \\
\ldots \ldots
\end{align*}
\]

There exists a countable sequence of points $\{P_i\}$ in $[0, 1]$ which converges to a subset of $C_0$ (the "base set" in $[0, 1]$) in a manner completely analogous to the convergence of $\{P_n\}$ in the basic example. Let $M' = C_0 \cup (\bigcup_{i=0}^\infty P_i)$ and let $f'$ be defined in a manner similar to $f$ in the basic example. Then $f'$ is a local contraction of a totally disconnected compact metric set onto itself which is not a local isometry. $M'$ and $f'$ in these last two examples can be extended as above to yield homeomorphisms.

**Bibliography**


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