ON COMBINATORIAL ARRANGEMENTS

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An arrangement of objects of v different varieties into b blocks (or sets) such that (i) no two blocks are identical (i.e. contain the same varieties), (ii) a variety occurs at most once in a block, (iii) any pair of varieties occurs together in λ blocks, $\lambda \neq 0$, (iv) every block contains k varieties, $k < v$, is called a balanced incomplete block design. These designs are of great use in applied statistics.

From (i), (ii), (iii), and (iv) it easily follows [1] that (v) all the varieties occur in the whole design an equal number of times, r, say, where $r = \lambda(v-1)/k-1$. But it is not difficult to see by constructing examples that the conditions (i), (ii), (iii), and (v) in general do not imply (iv).

About four years ago, Ryser [2] proved an interesting result (given here in an essentially equivalent form) that for symmetrical designs (i.e. designs in which $b = v$) conditions (i), (ii), (iii), and (v) imply (iv). In this note we give an extension of this result. To this end we first prove a general result on matrices given in Theorem 1. By column sum we shall mean the sum of the elements in a column.

THEOREM 1. *If two conformable matrices A, B (whose elements may belong to any given field) are such that*

(i) *column sums of C, where $C = AB$, are equal, to c say,*

(ii) *column sums of B are equal to, say b, then any column sum of A is c/b provided the rank of B is equal to the number of its rows.*

Let A and B be $m \times r$ and $r \times n$ matrices respectively. Since rank $B = r$, $n \geqq r$. Denote by 1 the unit element of the field. Pre-multiply the relation $AB = C$ by a row matrix composed of m 1's. On using (i) of this theorem we get

$$[s_1, s_2, \cdots, s_r]B = [c, c, \cdots, c]$$

where s_i denotes the sum of the ith column of A. Transposing the relation we get

$$B' \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_r \end{bmatrix} = \begin{bmatrix} c \\ c \\ \cdot \\ \cdot \\ \cdot \\ c \end{bmatrix}. \tag{1}$$

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The right-hand side of (1) is a column matrix of n c's. Now rank $B' = \text{rank } B = r$ and consequently r of the n rows of B' are linearly independent. Consider the corresponding r equations involved in (1). As the coefficient matrix is nonsingular, these equations have a unique solution $s_1, s_2, \cdots, s_r$. Since the sum of the coefficients in any of the r equations is b by condition (ii) of this theorem, the unique solution is easily seen to be $s_1 = s_2 = \cdots = s_r = c/b$. Of course, b cannot be the null element of the field—otherwise the rank of B is less than r. This completes the proof.

If the column sums of A are equal to a, then the equality of the column sums of any of B and C implies the equality of the column sums of the other and then $ab = c$—without any consideration of rank. This is easy to prove.

We now use Theorem 1 to establish a property of certain combinatorial arrangements. Suppose a "design" satisfies conditions (i) and (ii) of the first paragraph. List the v varieties in a column and the b blocks in a row. Construct an incidence matrix A of the design by putting 1 or 0 in the (ij) position of the matrix according as the ith variety occurs in the jth block or not. A design will be called nonsingular if rank A (or what is the same thing, rank AA') is equal to the number of blocks.

THEOREM 2. *If, in a nonsingular design, all the varieties appear an equal number of times, and if the total number of objects in all the blocks containing any particular variety is a constant, then any two blocks contain the same number of objects.*

Let each variety occur r times in the design. Then the column sums of A', the transpose of the incidence matrix, are all r. If $AA' = (\lambda_{ij})$, $i, j = 1, 2, \cdots, v$, then $\lambda_{ii} = r$, $i = 1, 2, \cdots, v$, and λ_{ij} is equal to the number of blocks in which ith and jth varieties occur together. Consider a fixed i. λ_{ij} then can be taken as the number of times the jth variety appears in the r blocks containing the ith variety. So $\sum_{j=1}^{v} \lambda_{ij}$, which is the column sum of the ith column of AA', is equal to a constant, say λ, by our assumption. Consequently taking $B = A'$ in Theorem I we get Theorem 2.

A particular case of this theorem immediately extends Ryser's result to symmetrical group divisible designs—a type of designs which are being extensively studied at present [**3**; **4**].

REFERENCES

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