ON COMBINATORIAL ARRANGEMENTS

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An arrangement of objects of $v$ different varieties into $b$ blocks (or sets) such that (i) no two blocks are identical (i.e. contain the same varieties), (ii) a variety occurs at most once in a block, (iii) any pair of varieties occurs together in $\lambda$ blocks, $\lambda \neq 0$, (iv) every block contains $k$ varieties, $k < v$, is called a balanced incomplete block design. These designs are of great use in applied statistics.

From (i), (ii), (iii), and (iv) it easily follows \[1\] that (v) all the varieties occur in the whole design an equal number of times, $r$, say, where $r = \lambda(v - 1)/k - 1$. But it is not difficult to see by constructing examples that the conditions (i), (ii), (iii), and (v) in general do not imply (iv).

About four years ago, Ryser \[2\] proved an interesting result (given here in an essentially equivalent form) that for symmetrical designs (i.e. designs in which $b = v$) conditions (i), (ii), (iii), and (v) imply (iv). In this note we give an extension of this result. To this end we first prove a general result on matrices given in Theorem 1. By column sum we shall mean the sum of the elements in a column.

**Theorem 1.** If two conformable matrices $A$, $B$ (whose elements may belong to any given field) are such that

(i) column sums of $C$, where $C = AB$, are equal, to $c$ say,
(ii) column sums of $B$ are equal to, say $b$, then any column sum of $A$ is $c/b$ provided the rank of $B$ is equal to the number of its rows.

Let $A$ and $B$ be $m \times r$ and $r \times n$ matrices respectively. Since rank $B = r$, $n \geq r$. Denote by 1 the unit element of the field. Pre-multiply the relation $AB = C$ by a row matrix composed of $m$ 1's. On using (i) of this theorem we get

$$[s_1, s_2, \ldots, s_r]B = [c, c, \ldots, c]$$

where $s_i$ denotes the sum of the $i$th column of $A$. Transposing the relation we get

$$B' \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{bmatrix} = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix}$$

Received by the editors December 1, 1953

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The right-hand side of (1) is a column matrix of \( n \) \( c \)'s. Now rank \( B' = \text{rank} \ B = r \) and consequently \( r \) of the \( n \) rows of \( B' \) are linearly independent. Consider the corresponding \( r \) equations involved in (1). As the coefficient matrix is nonsingular, these equations have a unique solution \( s_1, s_2, \ldots, s_r \). Since the sum of the coefficients in any of the \( r \) equations is \( b \) by condition (ii) of this theorem, the unique solution is easily seen to be \( s_1 = s_2 = \cdots = s_r = c/b \). Of course, \( b \) cannot be the null element of the field—otherwise the rank of \( B \) is less than \( r \). This completes the proof.

If the column sums of \( A \) are equal to \( a \), then the equality of the column sums of any of \( B \) and \( C \) implies the equality of the column sums of the other and then \( ab = c \)—without any consideration of rank. This is easy to prove.

We now use Theorem 1 to establish a property of certain combinatorial arrangements. Suppose a “design” satisfies conditions (i) and (ii) of the first paragraph. List the \( v \) varieties in a column and the \( b \) blocks in a row. Construct an incidence matrix \( A \) of the design by putting 1 or 0 in the \((ij)\) position of the matrix according as the \( i \)th variety occurs in the \( j \)th block or not. A design will be called nonsingular if rank \( A \) (or what is the same thing, rank \( AA' \)) is equal to the number of blocks.

**Theorem 2.** If, in a nonsingular design, all the varieties appear an equal number of times, and if the total number of objects in all the blocks containing any particular variety is a constant, then any two blocks contain the same number of objects.

Let each variety occur \( r \) times in the design. Then the column sums of \( A' \), the transpose of the incidence matrix, are all \( r \). If \( AA' = (\lambda_{ij}) \), \( i, j = 1, 2, \ldots, v \), then \( \lambda_{ii} = r, i = 1, 2, \ldots, v \), and \( \lambda_{ij} \) is equal to the number of blocks in which \( i \)th and \( j \)th varieties occur together. Consider a fixed \( i \). \( \lambda_{ij} \) then can be taken as the number of times the \( j \)th variety appears in the \( r \) blocks containing the \( i \)th variety. So \( \sum_{j=1}^{r} \lambda_{ij} \), which is the column sum of the \( i \)th column of \( AA' \), is equal to a constant, say \( \lambda \), by our assumption. Consequently taking \( B = A' \) in Theorem 1 we get Theorem 2.

A particular case of this theorem immediately extends Ryser's result to symmetrical group divisible designs—a type of designs which are being extensively studied at present [3; 4].

**References**

