A BOUND FOR A DETERMINANT WITH
DOMINANT MAIN DIAGONAL

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In [3, pp. 13–14], Hadamard proved that if a square matrix

\[(a_{ij})_{1 \leq i, j \leq n}\]

satisfies the inequalities

\[
\begin{align*}
  a_{ii} & \neq 0, \\
  \sigma_i \left| a_{kk} \right| & = \sum_{j \neq i} \left| a_{ij} \right|, \quad 0 \leq \sigma_i < 1 \quad (i = 1, 2, \ldots, n),
\end{align*}
\]

then the determinant of the matrix is different from 0. Upper and lower bounds have been given for the determinant of the matrix; see [4; 5; 6; 7]. In this article, improvements are given to the bounds of [5; 6]. It is remarkable that the bound of [5] could be improved by using nothing more complicated than ratios already introduced in that article: the bound for \(|\text{det } A| - |a_{11} \cdots a_{nn}|\) given there was asymptotically of the correct order in the \(l_j\) and \(r_j\). It is possible to improve the bounds of [4a] by the methods here given.


The following lemma is needed.

**Lemma.** Let \(A = (a_{ij})_{1 \leq i, j \leq n}\) be a matrix such that the relations (1) hold. Let \(b_{ij}\) be defined as \(a_{ij} - a_{1i}a_{ij}/a_{11}\). Then \((b_{ij})_{2 \leq i, j \leq n}\) has dominant main diagonal, and indeed the relations

\[
\begin{align*}
  \sigma_i \left| b_{ii} \right| & \geq \sum_{j > 1, j \neq i} \left| b_{ij} \right|, \\
  b_{ii} & \neq 0
\end{align*}
\]

hold for \(i = 2, 3, \ldots, n\).

This lemma states that the constant \(\sigma_i\) defined by the relation \(\sigma_i \left| b_{ii} \right| = \sum_{j > 1, j \neq i} \left| b_{ij} \right|\) is no greater than the corresponding constant \(\sigma_i\). The fact that \(\text{det } (b_{ij})\) is not 0 is established in [2].

**Proof.** The asserted inequalities follow from the hypotheses as follows.

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1 References 1a, 1b, 6, 7 and 2 added subsequently. The author wishes to thank the referee for calling his attention to the last.
Theorem 1. Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be a square matrix satisfying (1). Then a bound for \( \det A \) is
\[
| \det A | \geq | a_{11} | \prod_{i > 1} ( | a_{ii} | - l_i + L_i ),
\]
where \( l_i = \sum_{j<i} \sigma_j |a_{ij}|; L_i = |a_{ii}/a_{11}| \sum_{j > i} |a_{ij}|. \) Thus \( |a_{ii}| - l_i \) is automatically positive, and \( L_i \) is non-negative.

In both [5] and [6], a similar bound appears with \( L_i \) replaced by zero. Leaving this aside, the factor \( a_{ii} - l_i \) is still an improvement over the corresponding factor in [6], where \( l_i \) is replaced by the sum \( \sum_{j<i} |a_{ij}|, \) and is also an improvement over the corresponding factor in [5], where \( l_i \) is replaced by \( \sigma_i \sum_{j<i} |a_{ij}|, \) \( \sigma_i \) being the greatest of the \( \sigma_j \) with \( j \) not equal to \( i. \) Even in this form, the statement can be improved to read that \( \sigma_i \) is the greatest of the \( \sigma_j \) with \( j \) less than \( i. \)

This theorem is a special case of

Theorem 2. Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be a matrix such that relations (1) hold. A bound for \( | \det A | \) is given by the relation
\[
| \det A | \geq \prod_{i < k} ( | a_{ij} | - r_j + R_j ) \cdot | a_{kk} | \cdot \prod_{i > k} ( | a_{ij} | - l_j + L_j ),
\]
where \( r_j, l_j, R_j, L_j \) are defined by the relations
\[
\begin{align*}
r_j &= \sum_{i > j} \sigma_i |a_{ij}|, & L_j &= |a_{jk}/a_{kk}| \left( \sum_{i > j} |a_{kt}| + \sum_{t < k} |a_{kt}| \right),
R_j &= |a_{jk}/a_{kk}| \sum_{i < j} |a_{kt}|, & l_j &= \sum_{k \leq t < i} \sigma_t |a_{jt}|,
\end{align*}
\]
so that \( R_j \) and \( L_j \) are non-negative.

To prove Theorem 2, first reduce to 0 the nondiagonal elements of the \( k \)th row of \( A \) by adding an appropriate multiple of the \( k \)th column to the other columns. In the resulting matrix, call \( B \) the sub-matrix obtained by leaving out the \( k \)th row and the \( k \)th column. The \((i, j)\) element \( b_{ij} \) of \( B \) is \( a_{ij} - a_{ik} a_{jk}/a_{kk} (i \neq k, j \neq k). \) As an induc-
tion hypothesis, it is assumed that \( \det B \) satisfies the relation

\[
\det B \geq \prod_{j<k} (|b_{jj}| - r'_j + R'_j) \cdot |b_{k+1,k+1}|
\]

(3)

\[
\prod_{j>k+1} (|b_{jj}| - l'_j + L'_j),
\]

where \( R'_j, L'_j \) are non-negative, and \( r'_j, l'_j \) are defined by the relations

\[
r'_j = \sum_{l>j; t=k} \sigma_t |b_{jt}|, \quad l'_j = \sum_{k<i} \sigma_t |b_{jt}|.
\]

The theorem follows from a set of three estimates, which are established below by applying Lemma 1. The first estimate is that provided by the relations

\[
|b_{k+1,k+1}| \geq |a_{k+1,k+1}| - |a_{k+1,k}| \left( \sigma_k |a_{kk}| - \sum_{t \neq k, k+1} |a_{kt}| \right)
\]

\[
= |a_{k+1,k+1}| - \sigma_k |a_{k+1,k}| + |a_{k+1,k}/a_{kk}| \sum_{t \neq k, k+1} |a_{kt}|.
\]

The second estimate concerns the factor \(|b_{jj}| - l'_j\), when \( j \) exceeds \( k + 1 \):

\[
|b_{jj}| - l'_j \geq |a_{jj}| - |a_{jk}/a_{kk}| \left( \sigma_k |a_{kk}| - \sum_{t \neq j, k} |a_{kt}| \right)
\]

\[
- \sum_{k<j} \sigma_t |b_{jt}|
\]

\[
\geq |a_{jj}| - \sum_{k \geq l < j} \sigma_l |a_{jl}| + |a_{jk}/a_{kk}| \sum_{t \neq j, k} |a_{kt}|
\]

\[
- |a_{jk}/a_{kk}| \sum_{k<i} \sigma_i |a_{kt}|
\]

\[
\geq |a_{jj}| - \sum_{k \geq l < j} \sigma_l |a_{jl}| + |a_{jk}/a_{kk}| \left( \sum_{l>j} + \sum_{l<k} \right) |a_{kt}|.
\]

Last, it is necessary to estimate the factor \(|b_{jj}| - r'_j\) if \( j \) is less than \( k \):

\[
|b_{jj}| - r'_j \geq |a_{jj}| - \sigma_k |a_{jk}| + |a_{jk}/a_{kk}| \sum_{t \neq j, k} |a_{kt}| - \sum_{j<l; t \neq k} \sigma_t |b_{jt}|
\]

\[
\geq |a_{jj}| - \sum_{j<i} \sigma_t |a_{jt}|
\]

\[
+ |a_{jk}/a_{kk}| \left( \sum_{t \neq j, k} |a_{kt}| - \sum_{j<l; t \neq k} \sigma_t |a_{kt}| \right)
\]

\[
\geq |a_{jj}| - \sum_{j<i} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \sum_{k<i} |a_{kt}|.
\]
It will be observed that the above inequalities neglect certain terms which if retained would lead to an inequality slightly stronger than that of Theorem 2.

**Theorem 3.** Upper bounds for $|\det A|$ are the expressions obtained by mechanically reversing the four signs $-, +, -, +$, which appear on the right side of (2).

**References**


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