

A BOUND FOR A DETERMINANT WITH DOMINANT MAIN DIAGONAL

J. L. BRENNER

In [3, pp. 13-14], Hadamard proved that if a square matrix

$$(a_{ij})_{1 \leq i, j \leq n}$$

satisfies the inequalities

$$(1) \quad \begin{aligned} & a_{ii} \neq 0, \\ & \sigma_i |a_{kk}| = \sum_{j \neq i} |a_{ij}|, \quad 0 \leq \sigma_i < 1 \quad (i = 1, 2, \dots, n), \end{aligned}$$

then the determinant of the matrix is different from 0. Upper and lower bounds have been given for the determinant of the matrix; see [4; 5; 6; 7].¹ In this article, improvements are given to the bounds of [5; 6]. It is remarkable that the bound of [5] could be improved by using nothing more complicated than ratios already introduced in that article: the bound for $|\det A| - |a_{11} \cdots a_{nn}|$ given there was asymptotically of the correct order in the l_j and r_j . It is possible to improve the bounds of [1a] by the methods here given.

Certain generalizations of the present results appear in [1b]. Generalizations of some lemmas in [5] will appear elsewhere.

The following lemma is needed.

LEMMA. *Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix such that the relations (1) hold. Let b_{ij} be defined as $a_{ij} - a_{ii}a_{1j}/a_{11}$. Then $(b_{ij})_{2 \leq i, j \leq n}$ has dominant main diagonal, and indeed the relations*

$$\begin{aligned} \sigma_i |b_{ii}| &\geq \sum_{j > 1; j \neq i} |b_{ij}|, \\ b_{ii} &\neq 0 \end{aligned}$$

hold for $i = 2, 3, \dots, n$.

This lemma states that the constant σ'_i defined by the relation $\sigma'_i |b_{ii}| = \sum_{j > 1; j \neq i} |b_{ij}|$ is no greater than the corresponding constant σ_i . The fact that $\det (b_{ij})$ is not 0 is established in [2].

PROOF. The asserted inequalities follow from the hypotheses as follows.

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¹ References 1a, 1b, 6, 7 and 2 added subsequently. The author wishes to thank the referee for calling his attention to the last.

$$\begin{aligned} \sigma_i |b_{ii}| &\geq \sigma_i |a_{ii}| - \sigma_i |a_{i1}a_{1i}/a_{11}| \\ &\geq \sum_{j>1; j\neq i} |a_{1j}| + |a_{i1}/a_{11}| (|a_{11}| - \sigma_i |a_{1i}|) \\ &\geq \sum_{j>1; j\neq i} |a_{ij}| + |a_{i1}/a_{11}| \left(\sum_{j>1; j\neq i} |a_{1j}| + |a_{1i}| (1 - \sigma_i) \right) \\ &\geq \sum_{j>1; j\neq i} |b_{ij}|. \text{ Similarly, } |b_{ii}| > \sum_{j>1; j\neq i} |b_{ij}|. \end{aligned}$$

THEOREM 1. *Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a square matrix satisfying (1). Then a bound for $\det A$ is*

$$|\det A| \geq |a_{11}| \prod_{i>1} (|a_{ii}| - l_i + L_i),$$

where l_i is $\sum_{j<i} \sigma_j |a_{ij}|$; L_i is $|a_{i1}/a_{11}| \sum_{j>i} |a_{ij}|$. Thus $|a_{ii}| - l_i$ is automatically positive, and L_i is non-negative.

In both [5] and [6], a similar bound appears with L_i replaced by zero. Leaving this aside, the factor $a_{ii} - l_i$ is still an improvement over the corresponding factor in [6], where l_i is replaced by the sum $\sum_{j<i} |a_{ij}|$, and is also an improvement over the corresponding factor in [5], where l_i is replaced by $\sigma_t \sum_{j<i} |a_{ij}|$, σ_t being the greatest of the σ_j with j not equal to i . Even in this form, the statement can be improved to read that σ_t is the greatest of the σ_j with j less than i .

This theorem is a special case of

THEOREM 2. *Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix such that relations (1) hold. A bound for $|\det A|$ is given by the relation*

$$(2) \quad |\det A| \geq \prod_{j<k} (|a_{jj}| - r_j + R_j) \cdot |a_{kk}| \cdot \prod_{j>k} (|a_{jj}| - l_j + L_j),$$

where r_j, l_j, R_j, L_j are defined by the relations

$$\begin{aligned} r_j &= \sum_{t>j} \sigma_t |a_{jt}|, & L_j &= |a_{jk}/a_{kk}| \left(\sum_{t>j} |a_{kt}| + \sum_{t<k} |a_{kt}| \right), \\ R_j &= |a_{jk}/a_{kk}| \sum_{t<j} |a_{kt}|, & l_j &= \sum_{k \leq t < j} \sigma_t |a_{jt}|, \end{aligned}$$

so that R_j and L_j are non-negative.

To prove Theorem 2, first reduce to 0 the nondiagonal elements of the k th row of A by adding an appropriate multiple of the k th column to the other columns. In the resulting matrix, call B the submatrix obtained by leaving out the k th row and the k th column. The (i, j) element b_{ij} of B is $a_{ij} - a_{kj}a_{ik}/a_{kk}$ ($i \neq k, j \neq k$). As an induc-

tion hypothesis, it is assumed that $\det B$ satisfies the relation

$$(3) \quad \begin{aligned} |\det B| \geq & \prod_{j < k} (|b_{jj}| - r'_j + R'_j) \cdot |b_{k+1, k+1}| \\ & \cdot \prod_{j > k+1} (|b_{jj}| - l'_j + L'_j), \end{aligned}$$

where R'_j, L'_j are non-negative, and r'_j, l'_j are defined by the relations

$$r'_j = \sum_{t > j; t \neq k} \sigma_t |b_{jt}|, \quad l'_j = \sum_{k < l < j} \sigma_l |b_{jl}|.$$

The theorem follows from a set of three estimates, which are established below by applying Lemma 1. The first estimate is that provided by the relations

$$\begin{aligned} |b_{k+1, k+1}| & \geq |a_{k+1, k+1}| - |a_{k+1, k}| \left(\sigma_k |a_{kk}| - \sum_{t \neq k, k+1} |a_{kt}| \right) \\ & = |a_{k+1, k+1}| - \sigma_k |a_{k+1, k}| + |a_{k+1, k}/a_{kk}| \sum_{t \neq k, k+1} |a_{kt}|. \end{aligned}$$

The second estimate concerns the factor $|b_{jj}| - l'_j$, when j exceeds $k+1$:

$$\begin{aligned} |b_{jj}| - l'_j & \geq |a_{jj}| - |a_{jk}/a_{kk}| \left(\sigma_k |a_{kk}| - \sum_{t \neq j, k} |a_{kt}| \right) \\ & \quad - \sum_{k < t < j} \sigma_t |b_{jt}| \\ & \geq |a_{jj}| - \sum_{k \leq t < j} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \sum_{t \neq j, k} |a_{kt}| \\ & \quad - |a_{jk}/a_{kk}| \sum_{k < t < j} \sigma_t |a_{kt}| \\ & \geq |a_{jj}| - \sum_{k \leq t < j} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \left(\sum_{t > j} + \sum_{t < k} \right) |a_{kt}|. \end{aligned}$$

Last, it is necessary to estimate the factor $|b_{jj}| - r'_j$ if j is less than k :

$$\begin{aligned} |b_{jj}| - r'_j & \geq |a_{jj}| - \sigma_k |a_{jk}| + |a_{jk}/a_{kk}| \sum_{t \neq j, k} |a_{kt}| - \sum_{j < t; t \neq k} \sigma_t |b_{jt}| \\ & \geq |a_{jj}| - \sum_{j < t} \sigma_t |a_{jt}| \\ & \quad + |a_{jk}/a_{kk}| \left(\sum_{t \neq j, k} |a_{kt}| - \sum_{j < t; t \neq k} \sigma_t |a_{kt}| \right) \\ & \geq |a_{jj}| - \sum_{j < t} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \sum_{t < j} |a_{kt}|. \end{aligned}$$

It will be observed that the above inequalities neglect certain terms which if retained would lead to an inequality slightly stronger than that of Theorem 2.

THEOREM 3. *Upper bounds for $|\det A|$ are the expressions obtained by mechanically reversing the four signs $-$, $+$, $-$, $+$, which appear on the right side of (2).*

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STATE COLLEGE OF WASHINGTON