A BOUND FOR A DETERMINANT WITH DOMINANT MAIN DIAGONAL

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In [3, pp. 13–14], Hadamard proved that if a square matrix

$$(a_{ij})_{1\leqq i,j\leqq n}$$

satisfies the inequalities

$$(1) \qquad \begin{aligned} &a_{ii} \neq 0, \\ &\sigma_i \mid a_{kk} \mid = \sum_{j\neq i} \mid a_{ij} \mid, \ 0 \leqq \sigma_i < 1 \qquad (i = 1, 2, \cdots, n), \end{aligned}$$

then the determinant of the matrix is different from 0. Upper and lower bounds have been given for the determinant of the matrix; see [4; 5; 6; 7].[1] In this article, improvements are given to the bounds of [5; 6]. It is remarkable that the bound of [5] could be improved by using nothing more complicated than ratios already introduced in that article: the bound for $|\det A| - |a_{11} \cdots a_{nn}|$ given there was asymptotically of the correct order in the l_j and r_j. It is possible to improve the bounds of [1a] by the methods here given.

Certain generalizations of the present results appear in [1b]. Generalizations of some lemmas in [5] will appear elsewhere.

The following lemma is needed.

LEMMA. *Let $A = (a_{ij})_{1\leqq i,j\leqq n}$ be a matrix such that the relations* (1) *hold. Let b_{ij} be defined as $a_{ij} - a_{i1}a_{1j}/a_{11}$. Then $(b_{ij})_{2\leqq i,j\leqq n}$ has dominant main diagonal, and indeed the relations*

$$\sigma_i \mid b_{ii} \mid \geqq \sum_{j>1;\, j\neq i} \mid b_{ij} \mid,$$

$$b_{ii} \neq 0$$

hold for $i = 2, 3, \cdots, n$.

This lemma states that the constant σ_i' defined by the relation $\sigma_i' \, |b_{ii}| = \sum_{j>1;j\neq i} |b_{ij}|$ is no greater than the corresponding constant σ_i. The fact that det (b_{ij}) is not 0 is established in [2].

PROOF. The asserted inequalities follow from the hypotheses as follows.

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[1] References **1a, 1b, 6, 7** and **2** added subsequently. The author wishes to thank the referee for calling his attention to the last.

$$\begin{aligned}\sigma_i \left| b_{ii} \right| &\geqq \sigma_i \left| a_{ii} \right| - \sigma_i \left| a_{i1}a_{1i}/a_{11} \right| \\ &\geqq \sum_{j>1;\, j\neq i} \left| a_{1j} \right| + \left| a_{i1}/a_{11} \right| (\left| a_{11} \right| - \sigma_i \left| a_{1i} \right|) \\ &\geqq \sum_{j>1;\, j\neq i} \left| a_{ij} \right| + \left| a_{i1}/a_{11} \right| \left(\sum_{j>1;\, j\neq i} \left| a_{1j} \right| + \left| a_{1i} \right| (1 - \sigma_i) \right) \\ &\geqq \sum_{j>1;\, j\neq i} \left| b_{ij} \right|. \text{ Similarly, } \left| b_{ii} \right| > \sum_{j>1;\, j\neq i} \left| b_{ij} \right|.\end{aligned}$$

THEOREM 1. *Let* $A = (a_{ij})_{1\leqq i,j\leqq n}$ *be a square matrix satisfying* (1). *Then a bound for* det A *is*

$$\left| \det A \right| \geqq \left| a_{11} \right| \prod_{i>1} (\left| a_{ii} \right| - l_i + L_i),$$

where l_i *is* $\sum_{j<i} \sigma_j |a_{ij}|$; L_i *is* $|a_{i1}/a_{11}| \sum_{j>i} |a_{ij}|$. *Thus* $|a_{ii}| - l_i$ *is automatically positive, and* L_i *is non-negative.*

In both [5] and [6], a similar bound appears with L_i replaced by zero. Leaving this aside, the factor $a_{ii} - l_i$ is still an improvement over the corresponding factor in [6], where l_i is replaced by the sum $\sum_{j<i} |a_{ij}|$, and is also an improvement over the corresponding factor in [5], where l_i is replaced by $\sigma_t \sum_{j<i} |a_{ij}|$, σ_t being the greatest of the σ_j with j not equal to i. Even in this form, the statement can be improved to read that σ_t is the greatest of the σ_j *with* j *less than* i.

This theorem is a special case of

THEOREM 2. *Let* $A = (a_{ij})_{1\leqq i,j\leqq n}$ *be a matrix such that relations* (1) *hold. A bound for* $|\det A|$ *is given by the relation*

$$(2) \quad \left| \det A \right| \geqq \prod_{j<k} (\left| a_{jj} \right| - r_j + R_j) \cdot \left| a_{kk} \right| \cdot \prod_{j>k} (\left| a_{jj} \right| - l_j + L_j),$$

where r_j, l_j, R_j, L_j *are defined by the relations*

$$r_j = \sum_{t>j} \sigma_t \left| a_{jt} \right|, \qquad L_j = \left| a_{jk}/a_{kk} \right| \left(\sum_{t>j} \left| a_{kt} \right| + \sum_{t<k} \left| a_{kt} \right| \right),$$

$$R_j = \left| a_{jk}/a_{kk} \right| \sum_{t<j} \left| a_{kt} \right|, \qquad l_j = \sum_{k\leqq t<j} \sigma_t \left| a_{jt} \right|,$$

so that R_j *and* L_j *are non-negative.*

To prove Theorem 2, first reduce to 0 the nondiagonal elements of the kth row of A by adding an appropriate multiple of the kth column to the other columns. In the resulting matrix, call B the submatrix obtained by leaving out the kth row and the kth column. The (i, j) element b_{ij} of B is $a_{ij} - a_{kj}a_{ik}/a_{kk}$ $(i\neq k, j\neq k)$. As an induction hypothesis, it is assumed that det B satisfies the relation

$$
(3)\qquad \begin{aligned} |\det B| \geqq \prod_{j<k} (|b_{jj}| - r_j' + R_j') \cdot |b_{k+1,k+1}| \\ \cdot \prod_{j>k+1} (|b_{jj}| - l_j' + L_j'), \end{aligned}
$$

where R_j', L_j' are non-negative, and r_j', l_j' are defined by the relations

$$
r_j' = \sum_{t>j;\, t\neq k} \sigma_t |b_{jt}|, \qquad l_j' = \sum_{k<t<j} \sigma_t |b_{jt}|.
$$

The theorem follows from a set of three estimates, which are established below by applying Lemma 1. The first estimate is that provided by the relations

$$
\begin{aligned} |b_{k+1,k+1}| &\geqq |a_{k+1,k+1}| - |a_{k+1,k}| \Bigl(\sigma_k |a_{kk}| - \sum_{t\neq k,k+1} |a_{kt}|\Bigr) \\ &= |a_{k+1,k+1}| - \sigma_k |a_{k+1,k}| + |a_{k+1,k}/a_{kk}| \sum_{t\neq k,k+1} |a_{kt}|. \end{aligned}
$$

The second estimate concerns the factor $|b_{jj}| - l_j'$, when j exceeds $k+1$:

$$
\begin{aligned} |b_{jj}| - l_j' \geqq\ & |a_{jj}| - |a_{jk}/a_{kk}| \Bigl(\sigma_k |a_{kk}| - \sum_{t\neq j,k} |a_{kt}|\Bigr) \\ & - \sum_{k<t<j} \sigma_t |b_{jt}| \\ \geqq\ & |a_{jj}| - \sum_{k\leqq t<j} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \sum_{t\neq j,k} |a_{kt}| \\ & - |a_{jk}/a_{kk}| \sum_{k<t<j} \sigma_t |a_{kt}| \\ \geqq\ & |a_{jj}| - \sum_{k\leqq t<j} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \Bigl(\sum_{t>j} + \sum_{t<k}\Bigr) |a_{kt}|. \end{aligned}
$$

Last, it is necessary to estimate the factor $|b_{jj}| - r_j'$ if j is less than k:

$$
\begin{aligned} |b_{jj}| - r_j' \geqq\ & |a_{jj}| - \sigma_k |a_{jk}| + |a_{jk}/a_{kk}| \sum_{t\neq j,k} |a_{kt}| - \sum_{j<t;\, t\neq k} \sigma_t |b_{jt}| \\ \geqq\ & |a_{jj}| - \sum_{j<t} \sigma_t |a_{jt}| \\ & + |a_{jk}/a_{kk}| \Bigl(\sum_{t\neq j,k} |a_{kt}| - \sum_{j<t;\, t\neq k} \sigma_t |a_{kt}|\Bigr) \\ \geqq\ & |a_{jj}| - \sum_{j<t} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \sum_{t<j} |a_{kt}|. \end{aligned}
$$

It will be observed that the above inequalities neglect certain terms which if retained would lead to an inequality slightly stronger than that of Theorem 2.

THEOREM 3. *Upper bounds for* $|\det A|$ *are the expressions obtained by mechanically reversing the four signs* $-$, $+$, $-$, $+$, *which appear on the right side of* (2).

REFERENCES

1a. J. L. Brenner, *Une borne pour un déterminant avec diagonale majorante*, C.R. Acad. Sci. Paris vol. 238 (1954) pp. 555–556.

b. ———, *Bounds for determinants*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 452–454.

2. P. Furtwängler, Sitzungsber. Akad. Wiss. Wien IIa vol. 145 (1936) pp. 527–528.

3. J. Hadamard, *Leçons sur la propagation des ondes*, Paris, 1903.

4. Emilie V. Haynsworth, *Bounds for determinants with dominant main diagonal*, Duke Math. J. vol. 20 (1953) pp. 199–209.

5. A. M. Ostrowski, *Note on bounds for determinants with dominant principal diagonal*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 26–30.

6. G. B. Price, *Bounds for determinants with dominant principal diagonal*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 497–502.

7. Y. K. Wong, *An inequality for Minkowski matrices*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 137–141.

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