ON A GENERALIZATION OF AN INEQUALITY OF HARDY, LITTLEWOOD, AND PÓLYA[1]

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Two sets of n real numbers each are given:

$$(1) \qquad A = \{a_1 \leqq a_2 \leqq \cdots \leqq a_n\}, \qquad B = \{b_1 \leqq b_2 \leqq \cdots \leqq b_n\}.$$

These numbers will be regarded as distinct although two or more may be equal in value. Let n_i $(i=1, 2, \cdots, k)$ be any fixed set of k positive integers whose sum is n, let $C=\prod_{i=1}^{k} n_i!$ and let $R=n!/C$. We shall consider partitions $P^*=\{B_1^*, B_2^*, \cdots, B_k^*\}$ of the set B into disjoint subsets B_i^*, each B_i^* containing n_i elements of B $(i=1, 2, \cdots, k)$. By a partition here is meant an *ordered* k-tuplet of subsets, i.e. we consider two partitions P^*, P^{**} to be the same (or equal) if and only if $B_i^*=B_i^{**}$ $(i=1, 2, \cdots, k)$. The number of such partitions is given by the multinomial coefficient R. If we let B_1^1 dedenote the first n_1 elements of B, B_2^1 the next n_2 elements of B, $\cdots$, B_k^1 the last n_k elements of B, then $P^1=\{B_1^1, B_2^1, \cdots, B_k^1\}$ is a particular partition of B. Similarly if we let B_1^R denote the last n_1 elements of B, B_2^R the next n_2 elements of B, $\cdots$, B_k^R the first n_k elements of B, then $P^R=\{B_1^R, B_2^R, \cdots, B_k^R\}$ is another particular partition of B. Let $A_i^1=A_i$ $(i=1, 2, \cdots, k)$ be defined similarly for the set A, except that we shall regard the A_i as *ordered* subsets of A, the order being that given in (1). For convenience we define $N=\{1, 2, \cdots, n\}$ and the subsets $N_i^1=N_i$ $(i=1, 2, \cdots, k)$ exactly as was done for B.

The theorem that follows is concerned with the $n!$ cross products $\sum_{i=1}^{n} a_i b_{j_i}$ where $(j_1, j_2, \cdots, j_n)$ is a rearrangement of $(1, 2, \cdots, n)$. Corresponding to any fixed partition P^r $(r=1, 2, \cdots, R)$ we consider the set $V_r=\{v_{cr} \mid c=1, 2, \cdots, C\}$ of the C cross products obtained by associating the elements of B_i^r with A_i $(i=1, 2, \cdots, k)$, all possible rearrangements within the subsets B_i^r being allowed. The $n!$ cross products are thus divided in R sets with C cross products in each set.

Let P^r $(r=1, 2, \cdots, R)$ denote an arbitrary enumeration of the R partitions except that P^1 and P^R are defined above. We shall introduce (see *The partial ordering* below) a partial ordering (written $P^*>P^{**}$) among the partitions such that

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$$P^1 > P^r > P^R \qquad (r = 2, 3, \cdots, R-1). \tag{2}$$

THEOREM 1. *There exists a two-way table with R rows and C columns such that, for any pair* (r_1, r_2), *if* $P^{r_1} > P^{r_2}$ *then*

$$v_{cr_1} \geqq v_{cr_2} \qquad (c = 1, 2, \cdots, C). \tag{3}$$

COROLLARY 1. *For each* r $(r=1, 2, \cdots, R)$ *there exists a one-to-one pairing of the* v_{cr} *with the* v_{c1} *and also with the* v_{cR} *such that*

$$v_{c1} \geqq v_{cr} \geqq v_{cR} \qquad (c = 1, 2, \cdots, C). \tag{4}$$

PROOF. This follows immediately from (2) and Theorem 1, the columns of the two-way table providing the required one-to-one pairing.

COROLLARY 2. *For the special case* $n_i = 1$ $(i=1, 2, \cdots, k)$ *and hence* $n=k$ *we have for all rearrangements* $(j_1, j_2, \cdots, j_n)$ *of* $(1, 2, \cdots, n)$

$$\sum_{i=1}^{n} a_i b_i \geqq \sum_{i=1}^{n} a_i b_{j_i} \geqq \sum_{i=1}^{n} a_i b_{n-i+1}. \tag{5}$$

PROOF. If $n_i = 1$ $(i=1, 2, \cdots, k)$ then $C=1$, $R=n!$ and the result follows from Corollary 1. This is a known result given in [1, Theorem 368].

The partial ordering. Consider any partition P^r. We shall denote by a *left exchange* the operation of interchanging the positions in P^r of two elements $b_p \in B_s$ and $b_q \in B_t$ with say $p<q$ and having the following properties:

1. The two elements belong to different subsets in P^r, i.e. $s \neq t$.
2. The b's are going out of their natural order (1), i.e. $s<t$, and $p<q$.

Then define $P^{r_1} > P^{r_2}$ if there exists a nonempty ordered set (or product) of left exchanges that will take P^{r_1} into P^{r_2}. The properties of reflexiveness, transitivity, and antisymmetry are easily shown and hence the partitions form a *partially ordered system* (p.o.s.).

Some interesting properties of the p.o.s. thus formed were noted. For all values of n, k and $n_1, n_2, \cdots, n_k$ the p.o.s. thus formed appears to satisfy the constant chain condition, i.e. that all chains connecting the same 2 elements have the same length. For $k=2$ the p.o.s. forms a lattice while for $k>2$ it does not. These properties have not been proved (they are not used here) and therefore should be regarded as conjectures based on observed results. We also note that in all cases the replacement of every b_i by b_{n-i+1} reverses all inequalities and explains the symmetry that is always present.

We shall define a *unit left exchange* to be a *left exchange* of $b_p \in B_s$

and $b_q \in B_t$ such that none of the subsets $B_s, B_{s+1}, \cdots, B_t$ contains an element b_x with $p<x<q$. It will be seen below that this defines an immediate successor in the above p.o.s. If we define right exchange and unit right exchange similarly except that the b's go *into* their natural order (1), then a product of right exchanges and a unit right exchange correspond respectively to predecessor and immediate predecessor in the above p.o.s. We shall later have occasion to consider the subsets B_i $(i=1, 2, \cdots, k)$ as ordered. Then, clearly, a property of the unit exchanges is that they leave the ordering in each B_i unaltered.

To see that a unit left exchange corresponds to an immediate successor, consider any non-unit left exchange which takes P' into P''. Suppose the elements exchanged are $b_p \in B_s$ and $b_q \in B_t$. Then there is an element b_x in one of the sets $B_s, B_{s+1}, \cdots, B_t$ with $p<x<q$. If $b_x \notin B_s$, then define P^* as the partition P' with b_x and b_q interchanged; if $b_x \in B_s$, then define P^* as the partition P' with b_x and b_p interchanged. In either case we then have

$$P' > P^* > P''. \tag{6}$$

Conversely if the exchange connecting two partitions P' and P'' is a unit exchange then it is simple but tedious to show that there exists no partition P^* satisfying (6). We omit this proof.

Proof of Theorem 1. To prove (2) we shall first show that any given partition P^r can be reached by starting with P^1 and performing a sequence of left exchanges. It then follows from the previous paragraph that P^r can be reached by starting with P^1 and performing a sequence of *unit* left exchanges. Label each element b_j $(j=1, 2, \cdots, n)$ in P^1 as a left-mover, right-mover, or non-mover according to the direction it has to move in order to change P^1 to P^r. Let b_y be the first left-mover reading from left-to-right. Then there must be at least one right-mover to the left of b_y which is not in the same subset as b_y; let b_z be that particular one of these which is farthest to the right. Exchange b_y and b_z and then relabel all the b_j again as left-, right-, or non-movers. Repeat the procedure until P^r is formed. By this procedure left-movers (right-movers) continue moving to the left (right) until they become non-movers. It follows that each step of the procedure satisfies properties 1 and 2 and is therefore a left exchange. This proves the first part of (2); the second part follows similarly if we start with P^R and proceed by means of right exchanges to P^r.

Each of the $n!$ cross products $\sum_{i=1}^{n} a_i b_{j_i}$ can be uniquely identified by the associated permutation

$$\begin{pmatrix} 1, & 2, & \cdots, & n \\ j_1, & j_2, & \cdots, & j_n \end{pmatrix},$$

and the set of all these forms the symmetric group $S_n = S(N)$ on the n elements of N. Let $S(N_i)$ denote the symmetric group on the n_i elements of N_i $(i=1, 2, \cdots, k)$. Form the subgroup G of S_n

$$G = S(N_k) \times S(N_{k-1}) \times \cdots \times S(N_1) \tag{7}$$

which is the direct product of these k symmetric groups. It should be noted that the elements of G correspond to the C cross products v_{c1} of V_1 $(c=1, 2, \cdots, C)$.

We shall divide the group S_n into R cosets (corresponding to the rows of our table) modulo the subgroup G. Since G consists of all the permutations within the subsets $B_1^1, B_2^1, \cdots, B_k^1$, it is clear that each partition P^r will correspond to a row, say the rth row, of our table. The first row of our table will then consist of the cross products corresponding to the elements g_{c1} of G in any order except that, for convenience, we set g_{11} equal to the identity. The order within each of the remaining rows will then be determined by an induction on the p.o.s.

Suppose that the order of the cross products in the r_1th row, corresponding to P^{r_1}, has been determined; let g_{cr_1} denote the corresponding permutations. Suppose also that P^{r_2} is an immediate successor of P^{r_1} in the p.o.s. and that the associated unit exchange involves b_p and b_q with $p<q$. Then define the permutation

$$g_{cr_2} = (pq)g_{cr_1} \qquad (c = 1, 2, \cdots, C) \tag{8}$$

as the product of g_{cr_1} *by* the transposition $(p\ q)$. This determines the order of the cross product entries in the r_2th row, corresponding to P^{r_2}.

We first show that, for any pair (r_1, r_2), if $P^{r_1} > P^{r_2}$ then there exists a one-to-one pairing of $\{v_{cr_1}\}$ and $\{v_{cr_2}\}$ such that

$$v_{cr_1} \geqq v_{cr_2} \qquad (c = 1, 2, \cdots, C). \tag{9}$$

It suffices to assume that P^{r_2} is an immediate successor of P^{r_1} in the p.o.s. Let $b_p \in B_s$ and $b_q \in B_t$ in P^{r_1} be the elements of B exchanged. Assume $p<q$. Then, by (1), $b_p \leqq b_q$ and by property 2 we have $s<t$. Hence for any fixed c it follows that the coefficients of b_p and b_q in v_{cr_1}, say a_{cp} and a_{cq}, belong to the distinct subsets A_s and A_t, respectively. Hence $a_{cp} \leqq a_{cq}$. It is easy to see that for each c the above exchange affects only terms involving b_p and b_q and in fact that

$$v_{cr_1} - v_{cr_2} = (a_{cq} - a_{cp})(b_q - b_p) \geqq 0 \qquad (c = 1, 2, \cdots, C). \tag{10}$$

This proves (9).

To complete the proof of the theorem we need only consider partitions of P^{r_2} which have more than one immediate predecessor, say P^{r_1} and P^{r_0}, in the p.o.s. and show that the above ordering of the r_2th row is the same regardless of which predecessor is used. By (2) we can refer these two methods to the common starting point P^1 and write for one method

$$\overset{1}{g}_{cr_2} = g_1 g_{c1} \qquad (c = 1, 2, \cdots), \tag{11}$$

and for the other

$$\overset{0}{g}_{cr_2} = g_0 g_{c1} \qquad (c = 1, 2, \cdots, C). \tag{12}$$

It follows from (11) and (12) that g_0 and g_1 are in the same coset. The problem is to show that $g_0 = g_1$. If g_{11} is the identity it is sufficient by (11) and (12) to show that $\overset{1}{g}_{1r_2} = \overset{0}{g}_{1r_2}$.

Up to this point the order within the subsets B_i^* in $P^* = \{B_1^*, B_2^*, \cdots, B_k^*\}$ has not been considered but by taking this order into account we can identify the individual cross products in each row and hence also the corresponding permutations. For example if the partition with ordering within subsets is $\{(b_1); (b_3, b_2)\}$, then the corresponding cross product is $a_1b_1 + a_2b_3 + a_3b_2$ and the corresponding permutation is

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23).$$

Let us start at the top of our p.o.s. with $P^1 = \{B_1^1, B_2^1, \cdots, B_k^1\}$, each subset B_i^1 being in natural order (1), and consider the effects of the various exchanges as we travel down different "paths" leading to P^{r_2}. It was noted above that each unit exchange leaves the order within the subsets unaltered. Then $\overset{1}{g}_{1r_2}$ and $\overset{0}{g}_{1r_2}$ correspond to the same partition with the *same ordering within subsets* and hence they are equal. This completes the proof of the theorem.

Corollary 3. *If in addition to* (1) *we have no two b's equal and*

$$a_i < a_{i+1} \qquad \left(i = n_1, n_1 + n_2, \cdots, \sum_{i=1}^{k} n_i\right), \tag{13}$$

then Theorem 1 *holds with strict inequality for each c* ($c = 1, 2, \cdots, C$).

Proof. Under the added hypothesis strict inequality holds for both factors in (10) and the result follows.

This problem arose in connection with a statistical problem [2] involving a sequential multiple decision procedure for ranking parameters. Two corollaries of particular interest in [2] will now be given.

COROLLARY 4. *If* $P^{r_1} > P^{r_2}$, *then for each* c $(c=1, 2, \cdots, C)$ *we can write* $v_{cr_1} - v_{cr_2}$ *as a sum of non-negative terms in the form*

$$v_{cr_1} - v_{cr_2} = \sum (b_q - b_p)(a_w - a_u) \tag{14}$$

where

$$q \geqq w > u \geqq p. \tag{15}$$

PROOF. Label each element b_j $(j=1, 2, \cdots, k)$ in P^{r_1} as a left-, right- or non-mover according to its position in P^{r_2}. Proceed as in the proof of (2) above to find a sequence of left exchanges leading from P^{r_1} to P^{r_2}. To each exchange in turn we can apply the same argument as was used to obtain (10). To prove (15) we refer to the property of this method, noted in the proof of (2), that left-movers (right-movers) continue to move to the left (right). Since b_q is a left-mover and a_w is its "temporary position" it follows that $q \geqq w$. Similarly, b_p being a right-mover, $p \leqq u$. By properties 1 and 2 we have that a_u and a_w belong to different subsets, say $a_u \in A_s$ and $a_w \in A_t$, and that $s < t$. It follows that $u < w$.

For the last corollary we consider the special case $k=2$ which is the case that arises in the application [2] and define $b_0 = -\infty$ and $b_{n+1} = +\infty$. We shall assume in addition to (1) that

$$\begin{aligned} b_{n_1-m_1} < b_{n_1-m_1+1} = b_{n_1-m_1+2} = \cdots = b_{n_1} = b_{n_1+1} = \cdots \\ = b_{n_1+m_2} < b_{n_1+m_2+1} \end{aligned} \tag{16}$$

where $1 \leqq m_1 \leqq n_1$ and $0 \leqq m_2 \leqq n_2$.

COROLLARY 5. (a) *If* m_1 *and* m_2 *are defined by* (16), *then the two-way table of Theorem* 1 *will contain* $C_{m_2}^{m_1+m_2}$ *rows, including the first row, such that entries in the same column are equal.* (b) *If* $m_1 + m_2 < n$ *and*

(i) $a_{n_1} < a_{n_1+1}$,

(ii) *not all the a's are equal,*

then comparing any one of the remaining rows with the first row we will have in cases (i) *and* (ii) *respectively*

(i) *strict inequality in each column,*

(ii) *strict inequality in at least one column.*

PROOF. (a) By Theorem 1, the first and largest row is obtained by associating with the fixed set $A_2 = \{a_{n_1+1}, a_{n_1+2}, \cdots, a_n\}$ the set or combination $B_2^1 = \{b_{n_1+1}, b_{n_1+2}, \cdots, b_n\}$ as coefficients. Clearly, exchanging any b in B_2^1 with another b equal in value from B_1^1 will not affect any of the cross products v_{c1} ($c=1, 2, \cdots, C$). Hence by (16) there are $C_{m_2}^{m_1+m_2}$ rows, including the first row, that are equivalent.

(b) If $m_1+m_2<n$, then not all the b's are equal and there is at least one more row in our two-way table not included in (a), i.e. there is at least one more partition P^{r_0} not included in (a). We can assume, for simplicity, that the partition P^{r_0} differs from P^1 by an exchange of b_p and b_q where either $p \leqq n_1 - m_1$ and $q > n_1$ or $p \leqq n_1$ and $q > n_1 + m_2$. This assumption is permitted since any partition P^{r_0} not in (a) differs from P^1 by a product of exchanges, at least one of which is of the above type. Then $b_p < b_q$. For each c ($c=1, 2, \cdots, C$) we exchange b_p and b_q in v_{c1} to form v_{cr_0}. If we let a_{cp} and a_{cq} denote the coefficients of b_p and b_q respectively in v_{c1}, then clearly, for case (i), $a_{cq} > a_{cp}$ for every c and, for case (ii), $a_{cq} \geqq a_{cp}$ for every c with strict inequality holding for at least one c. Then we obtain as in (10) above

$$(17) \qquad v_{c1} - v_{cr_0} = (a_{cq} - a_{cp})(b_q - b_p) \geqq 0 \qquad (c = 1, 2, \cdots, C).$$

For case (i) we have strict inequality for each c and for case (ii) we have strict inequality for at least one c.

The ordering of row r_0 described above must be the same as the ordering in Theorem 1 since the left exchange of b_p with b_q can be expressed as a product of unit left exchanges.

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