NOTE ON THE FIRST CESARO MEAN OF THE DERIVED FOURIER SERIES

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1. Let \( f(t) \) be integrable \( L \) in \( (-\pi, \pi) \) and periodic with period \( 2\pi \) and let

\[
f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t).
\]

Then the differentiated series of \( \sum_{n=1}^{\infty} A_n(t) \) is

\[
\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t).
\]

We write

\[
\psi(t) = f(x + t) - f(x - t), \quad g(t) = \frac{f(x + t) - f(x - t)}{2 \sin \left(\frac{t}{2}\right)}.
\]

Let \( S_n(x) \) be the \( nth \) partial sum and \( t_n(x) \) the first arithmetic mean of the series (1), where

\[
S_n(x) = \sum_{r=1}^{n} rB_r(x),
\]

\[
t_n(x) = \frac{S_1 + S_2 + \cdots + S_n}{n}.
\]

The object of the present note is to prove

**Theorem 1.** If

\[
\int_{0}^{t} |g(u) - c| \, du = o(t) \quad \text{when } t \to 0,
\]

then \( t_n = o(\log n) \).

2. Proof of Theorem 1. We have

\[
rB_r(x) = \frac{1}{\pi} \int_{0}^{\pi} \left\{ f(x + t) - f(x - t) \right\} r \sin rtdt
\]

\[
= \frac{1}{\pi} \int_{0}^{\pi} \psi(t) r \sin rtdt.
\]

Received by the editors October 26, 1953.

\^Throughout the present note there is the tacit assumption that \( g(t) \) is \( L \) in \( (0, \pi) \) and \( c \) a function of \( x \).
Hence

\[
S_n(x) = \frac{1}{\pi} \int_0^\pi \psi(t) dt \sum_{r=1}^n r \sin rt
\]

\[
= - \frac{1}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} \left\{ \frac{1}{2} + \sum_{k=1}^n \cos rt \right\} dt
\]

\[
= - \frac{1}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} \left\{ \frac{\sin(n+1/2)t}{2 \sin(t/2)} \right\} dt.
\]

Now from (4) and (7) we have

\[
t_n(x) = \frac{1}{n} \sum_{k=1}^n S_k
\]

\[
= - \frac{1}{\pi n} \int_0^\pi \psi(t) \frac{d}{dt} \left\{ \frac{1}{2 \sin(t/2)} \sum_{k=1}^n \sin \left( k + \frac{1}{2} \right) t \right\} dt
\]

\[
= - \frac{1}{\pi n} \int_0^\pi \psi(t) \frac{d}{dt} \left[ \frac{\cos(t/2) \{ \cos(t/2) - \cos(n+1/2)t \}}{4 \sin^2(t/2)} + \frac{1}{2} \left\{ \frac{\sin(n+1/2)t}{2 \sin(t/2)} - \frac{1}{2} \right\} \right] dt
\]

\[
= - \frac{1}{\pi n} \int_0^\pi \psi(t) \frac{d}{dt} \left\{ \frac{\cos t}{4 \sin^2(t/2)} - \frac{\cos(n+1)t}{4 \sin^2(t/2)} \right\} dt
\]

\[
= \frac{1}{\pi n} \int_0^\pi g(t) \cos(t/2) \frac{1 - \cos(n+1)t}{2 \sin^2(t/2)} dt
\]

\[
- \frac{n+1}{n} \frac{1}{\pi} \int_0^\pi g(t) \frac{\sin(n+1)t}{2 \sin(t/2)} dt
\]

\[
= \frac{1}{\pi n} \int_0^\pi g^*(t) \cos(t/2) \left( \frac{\sin^2((n+1)t/2)}{\sin^2(t/2)} \right) dt
\]

\[
- \frac{n+1}{n} \frac{1}{\pi} \int_0^\pi g^*(t) \frac{\sin(n+1)t}{2 \sin(t/2)} dt
\]

\[
+ \frac{c}{\pi n} \int_0^\pi \cos(t/2) \left( \frac{\sin^2((n+1)t/2)}{\sin^2(t/2)} \right) dt
\]

\[
- \frac{c}{\pi} \frac{n+1}{n} \int_0^\pi \frac{\sin(n+1)t}{2 \sin(t/2)} dt \quad \text{(where } g(t) - c = g^*(t) \text{)}
\]

\[
= I_1 + I_2 + I_3 + I_4, \text{ say.}
\]
Now using the proof of the Fejér-Lebesgue theorem for summability \((c, 1)\) of the Fourier series\(^2\) with \(g^*(t)\) in place of \(\phi(t)\), it can be proved that

\[
I_1 = o(1).
\]

Using the argument similar to that employed in the proof of a well-known result\(^3\) and the Condition (5) of Theorem 1, we have

\[
I_2 = o(\log n).
\]

Again

\[
|I_3| = \frac{|c|}{\pi n} \int_0^\pi \cos (t/2) \frac{\sin^2 ((n + 1)t/2)}{\sin^2 (t/2)} \, dt
\]

\[
< o(1) + \frac{|c|}{\pi n} \int_0^\pi \frac{\sin^2 ((n + 1)t/2)}{t^2} \, dt
\]

\[
= o(1) + \frac{|c|}{\pi} \frac{n + 1}{n} \int_0^{(n+1)\pi/2} \frac{\sin^2 \theta}{\theta^2} \, d\theta
\]

\[
\rightarrow \frac{|c|}{\pi} \int_0^\infty \frac{\sin^2 \theta}{\theta^2} \, d\theta
\]

\[
= |c|/2.
\]

Lastly

\[
I_4 = -\frac{c}{\pi} \frac{n + 1}{n} \int_0^\pi \sin (n + 1)t \frac{\sin (t/2)}{\sin (t/2)} \, dt
\]

\[
= -\frac{c}{\pi} \frac{n + 1}{n} \left[ \int_0^{(n+1)\pi} \frac{\sin \theta}{\theta} \, d\theta + o(1) \right]
\]

\[
\rightarrow -\frac{c}{\pi} \int_0^\infty \frac{\sin \theta}{\theta} \, d\theta
\]

\[
= -c/2.
\]

Hence by \((7)\), \((9)\), \((10)\), \((11)\), and \((12)\), we have

\[
t_n = o(\log n).
\]

3. Using relation \((13)\) above we prove

**Theorem 2.** If

\[
g(t) \rightarrow c,
\]

when \(t \rightarrow 0\)


\(^3\) *Fourier series* by G. H. Hardy and W. W. Rogosinski, Theorem 64, p. 49.
then
\[ \sum_{r=1}^{\infty} \frac{rB_r(x)}{\log (r + 1)} \text{ is summable } (c, 1). \]

To prove Theorem 2 we require the following lemmas:

**Lemma I.** If \( \sum_{r=1}^{\infty} U_r \) is summable \((c, k+1)\), where \( k > -1 \), then a necessary and sufficient condition that it should be summable \((c, k)\) is that
\[ \sum_{r=1}^{n} A_r \cdot r = o(n^{k+1}); \text{ where } A_m = \frac{(k+1)(k+2) \cdots (k+m)}{m!}. \]

**Lemma II.** If \( \int_0^t |g^*(u)| \, du = o(t) \), then \( S_n(x) = o(n) \).

**Proof of Lemma II.**
\[ S_n(x) = -\frac{1}{\pi} \int_0^x \psi(t) \frac{d}{dt} \left\{ \frac{\sin (n+1/2) t}{2 \sin (t/2)} \right\} dt \]
(14)
\[ = \frac{1}{\pi} \int_0^x g(t) \frac{\cos (t/2) \sin (n+1/2) t}{2 \sin (t/2)} dt \]
\[ - \frac{1}{\pi} \left( n + \frac{1}{2} \right) \int_0^x g(t) \cos \left( n + \frac{1}{2} \right) t dt \]
\[ = o(\log n) + o(n) = o(n), \]
using the condition of Lemma II.

**Proof of Theorem 2.** If \( g(t) \to c \) when \( t \to 0 \), then \( \sum_{r=1}^{\infty} rB_r(x) \) is summable \((c, 2)\)^6 hence in view of the case \( k=1 \) of Lemma I above, it is enough for the proof of Theorem 2 to show that
\[ \sum_{r=1}^{n} A'_r \cdot rB_r(x) = o(n^2). \]
\[ \sum_{r=1}^{n} A'_r rB_r(x) \frac{r}{\log (r + 1)} = o(n^2). \]

Applying Abel's transformation twice, we find
\[ \sum_{r=1}^{n} A'_r rB_r(x) \frac{r}{\log (r + 1)} = \sum_{r=1}^{n-2} (r+1)l_r \Delta^2 \left( A'_r \frac{r}{\log (r + 1)} \right) \]
\[ + nl_{n-1} \Delta \left( \frac{n}{\log n} \right) + S_n \frac{n+1}{\log (n+1)} \]
\[ = T_1 + T_2 + T_3, \text{ say.} \]

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Straightforward calculations give

\[(16) \quad \Delta \left( \frac{r}{\log r} \right) = O \left( \frac{1}{\log r} \right) \]

and

\[(17) \quad \Delta^2 \left( A'_{n-r} \frac{r}{\log (r+1)} \right) = O \left( \frac{1}{\log r} \right). \]

Using (13) and (16) we have

\[T_2 = o(n), \]

and by Lemma II we find

\[T_3 = o \left( \frac{n^2}{\log n} \right). \]

Now by (13) and (17) we have

\[T_1 = \sum_{r=1}^{n-2} (r + 1) o(\log r) \ O \left( \frac{1}{\log r} \right) = o(n^2). \]

Thus

\[\sum_{r=1}^{n} A'_{n-r} B_r(x) \frac{r}{\log (r+1)} = o(n^2) \]

which completes the proof of Theorem 2.

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