NOTE ON THE FIRST CESARO MEAN OF THE DERIVED FOURIER SERIES

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1. Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π and let

$$f(t) \sim \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_1^\infty A_n(t).$$

Then the differentiated series of $\sum_1^\infty A_n(t)$ is

$$\sum_1^\infty n(b_n \cos nt - a_n \sin nt) = \sum_1^\infty nB_n(t). \tag{1}$$

We write

$$\psi(t) = f(x + t) - f(x - t), \qquad g(t) = \frac{f(x + t) - f(x - t)}{2 \sin (t/2)} . \tag{2}$$

Let $S_n(x)$ be the nth partial sum and $t_n(x)$ the first arithmetic mean of the series (1), where

$$S_n(x) = \sum_{r=1}^{n} rB_r(x), \tag{3}$$

$$t_n(x) = (S_1 + S_2 + \cdots + S_n)/n. \tag{4}$$

The object of the present note is to prove

THEOREM 1. *If*[1]

$$\int_0^t | g(u) - c | \, du = o(t) \qquad \textit{when } t \to 0, \tag{5}$$

then $t_n = o(\log n)$.

2. PROOF OF THEOREM 1. We have

$$\begin{aligned} rB_r(x) &= \frac{1}{\pi} \int_0^\pi \{f(x + t) - f(x - t)\} r \sin rt dt \\ &= \frac{1}{\pi} \int_0^\pi \psi(t) r \sin rt dt. \end{aligned} \tag{6}$$

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[1] Throughout the present note there is the tacit assumption that $g(t)$ is L in $(0, \pi)$ and c a function of x.

Hence

$$S_n(x) = \frac{1}{\pi}\int_0^\pi \psi(t)dt \sum_{r=1}^{n} r \sin rt$$

$$(7) \qquad = -\frac{1}{\pi}\int_0^\pi \psi(t) \frac{d}{dt}\left\{\frac{1}{2} + \sum_1^n \cos rt\right\} dt$$

$$= -\frac{1}{\pi}\int_0^\pi \psi(t) \frac{d}{dt}\left\{\frac{\sin(n+1/2)t}{2\sin(t/2)}\right\} dt.$$

Now from (4) and (7) we have

$$t_n(x) = \frac{1}{n}\sum_{k=1}^{n} S_k$$

$$= -\frac{1}{\pi n}\int_0^\pi \psi(t) \frac{d}{dt}\left\{\frac{1}{2\sin(t/2)} \sum_{k=1}^{n} \sin\left(k+\frac{1}{2}\right)t\right\} dt$$

$$= -\frac{1}{\pi n}\int_0^\pi \psi(t) \frac{d}{dt}\left[\frac{\cos(t/2)\{\cos(t/2) - \cos(n+1/2)t\}}{4\sin^2(t/2)} + \frac{1}{2}\left\{\frac{\sin(n+1/2)t}{2\sin(t/2)} - \frac{1}{2}\right\}\right] dt$$

$$= -\frac{1}{\pi n}\int_0^\pi \psi(t) \frac{d}{dt}\left\{\frac{\cos t}{4\sin^2(t/2)} - \frac{\cos(n+1)t}{4\sin^2(t/2)}\right\} dt$$

$$(8) \qquad = \frac{1}{\pi n}\int_0^\pi g(t)\cos(t/2)\, \frac{1-\cos(n+1)t}{2\sin^2(t/2)}\, dt - \frac{n+1}{n}\,\frac{1}{\pi}\int_0^\pi g(t)\, \frac{\sin(n+1)t}{2\sin(t/2)}\, dt$$

$$= \frac{1}{\pi n}\int_0^\pi g^*(t)\cos(t/2)\left(\frac{\sin^2((n+1)t/2)}{\sin^2(t/2)}\right) dt - \frac{n+1}{n}\,\frac{1}{\pi}\int_0^\pi g^*(t)\, \frac{\sin(n+1)t}{2\sin(t/2)}\, dt$$

$$+ \frac{c}{\pi n}\int_0^\pi \cos(t/2)\left(\frac{\sin^2((n+1)t/2)}{\sin^2(t/2)}\right) dt - \frac{c}{\pi}\,\frac{n+1}{n}\int_0^\pi \frac{\sin(n+1)t}{2\sin(t/2)}\, dt \quad (\text{where } g(t) - c = g^*(t))$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

Now using the proof of the Fejér-Lebesgue theorem for summability $(c, 1)$ of the Fourier series[2] with $g^*(t)$ in place of $\phi(t)$, it can be proved that

$$I_1 = o(1). \tag{9}$$

Using the argument similar to that employed in the proof of a well-known result[3] and the Condition (5) of Theorem 1, we have

$$I_2 = o(\log n). \tag{10}$$

Again

$$\begin{aligned} |I_3| &= \frac{|c|}{\pi n}\int_0^{\pi} \cos(t/2)\,\frac{\sin^2((n+1)t/2)}{\sin^2(t/2)}\,dt \\ &< o(1) + \frac{|c|}{\pi n}\int_0^{\pi} \frac{\sin^2((n+1)t/2)}{t^2}\,dt \\ &= o(1) + \frac{|c|}{\pi}\,\frac{n+1}{n}\int_0^{(n+1)\pi/2} \frac{\sin^2\theta}{\theta^2}\,d\theta \\ &\to \frac{|c|}{\pi}\int_0^{\infty} \frac{\sin^2\theta}{\theta^2}\,d\theta \\ &= |c|/2. \end{aligned} \tag{11}$$

Lastly

$$\begin{aligned} I_4 &= -\frac{c}{\pi}\,\frac{n+1}{n}\int_0^{\pi} \frac{\sin(n+1)t}{\sin(t/2)}\,dt \\ &= -\frac{c}{\pi}\,\frac{n+1}{n}\left[\int_0^{(n+1)\pi} \frac{\sin\theta}{\theta}\,d\theta + o(1)\right] \\ &\to -\frac{c}{\pi}\int_0^{\infty} \frac{\sin\theta}{\theta}\,d\theta \\ &= -c/2. \end{aligned} \tag{12}$$

Hence by (7), (9), (10), (11), and (12), we have

$$t_n = o(\log n). \tag{13}$$

3. Using relation (13) above we prove

THEOREM 2. *If*

$$g(t) \to c, \qquad \textit{when } t \to 0$$

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[2] *The theory of functions*, E. C. Titchmarsh, 1939, p. 415.

[3] *Fourier series* by G. H. Hardy and W. W. Rogosinski, Theorem 64, p. 49.

then

$$\sum_{r=1}^{\infty} \frac{rB_r(x)}{\log (r+1)} \qquad \textit{is summable } (c, 1).$$

To prove Theorem 2 we require the following lemmas:

LEMMA I.[4] *If* $\sum_{r=1}^{\infty} U_r$ *is summable* $(c, k+1)$, *where* $k > -1$, *then a necessary and sufficient condition that it should be summable* (c, k) *is that*

$$\sum_{r=1}^{n} A_{n-r}^{k} U_r \cdot r = o n^{k+1}; \textit{ where } A_m^k = \frac{(k+1)(k+2)\cdots(k+m)}{m!}.$$

LEMMA II. *If* $\int_0^t |g^*(u)| du = o(t)$, *then* $S_n(x) = o(n)$.

PROOF OF LEMMA II.

$$S_n(x) = -\frac{1}{\pi}\int_0^{\pi} \psi(t) \frac{d}{dt}\left\{\frac{\sin (n+1/2)t}{2 \sin (t/2)}\right\} dt$$

$$= \frac{1}{\pi}\int_0^{\pi} g(t) \frac{\cos (t/2) \sin (n+1/2)t}{2 \sin (t/2)} dt \tag{14}$$

$$- \frac{1}{\pi}\left(n+\frac{1}{2}\right)\int_0^{\pi} g(t) \cos \left(n+\frac{1}{2}\right) t dt$$

$$= o(\log n) + o(n) = o(n),$$

using the condition of Lemma II.

PROOF OF THEOREM 2. If $g(t) \rightarrow c$ when $t \rightarrow 0$, then $\sum_{r=1}^{\infty} rB_r(x)$ is summable $(c, 2)$;[5] hence in view of the case $k=1$ of Lemma I above, it is enough for the proof of Theorem 2 to show that

$$\sum_{r=1}^{n} A'_{n-r} r B_r(x) \frac{r}{\log (r+1)} = o(n^2). \tag{15}$$

Applying Abel's transformation twice, we find

$$\sum_{r=1}^{n} A'_{n-r} r B_r(x) \frac{r}{\log (r+1)} = \sum_{r=1}^{n-2} (r+1) t_r \Delta^2 \left(A'_{n-r} \frac{r}{\log (r+1)}\right)$$

$$+ n t_{n-1} \Delta\left(\frac{n}{\log n}\right) + S_n \frac{n+1}{\log (n+1)}$$

$$= T_1 + T_2 + T_3, \text{ say.}$$

[4] G. H. Hardy, *Divergent series*, 1949, Theorem 65, p. 122.

[5] Zygmund, *Trigonometrical series*, 1935, p. 55.

Straightforward calculations give

$$\Delta\left(\frac{r}{\log r}\right) = O\left(\frac{1}{\log r}\right) \tag{16}$$

and

$$\Delta^2\left(A'_{n-r} \frac{r}{\log (r+1)}\right) = O\left(\frac{1}{\log r}\right). \tag{17}$$

Using (13) and (16) we have

$$T_2 = o(n),$$

and by Lemma II we find

$$T_3 = o\left(\frac{n^2}{\log n}\right).$$

Now by (13) and (17) we have

$$T_1 = \sum_{r=1}^{n-2} (r+1)o(\log r)\ O\left(\frac{1}{\log r}\right) = o(n^2).$$

Thus

$$\sum_{r=1}^{n} A'_{n-r} r B_r(x) \frac{r}{\log (r+1)} = o(n^2)$$

which completes the proof of Theorem 2.

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