EXTENSIONS OF THE LAPLACE METHOD[1]

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The Laplace method [1][3] is concerned with the evaluation of the integral $\int_0^a \exp[-h\phi(t)]dt$ $(a>0)$ as $h\to+\infty$. Under suitable conditions on $\phi(t)$ it can be shown that

$$\int_0^a \exp[-h\phi(t)]dt \sim \int_0^a \exp\left[-\frac{1}{2}h\phi''(0)t^2\right]dt \sim \left(\frac{2\pi}{h\phi''(0)}\right)^{1/2}.$$

We write $A\sim B$ to indicate that $(A-B)/B\to 0$. In this paper the integral

$$J=\int_0^a \exp[-h\phi(t)+k\psi(t)]dt \tag{1}$$

is considered as $h, k\to+\infty$ and $a\to 0+$. Fulks [2] considered the case where a is fixed while $h, k\to+\infty$ and derived explicit asymptotic expressions assuming suitable conditions on $\phi(t)$, $\psi(t)$ and also assuming that, as $h, k\to+\infty$, $k=o(h)$; interesting applications were also given to the incomplete gamma function. Under similar conditions we now derive expressions asymptotic to J as $h, k\to+\infty$ and in addition as $a\to 0+$, a case not considered in [2].

It is assumed that the following conditions are satisfied at various parts of the discussion.

(A) $\phi(t)$ is positive and nondecreasing in $0\leqq t<c$. $\psi(t)$ is real in $0\leqq t<c$.

(B) $\phi(t)\in C^3$, $\psi(t)\in C^2$ $(0\leqq t<c)$.

(C) $\phi(t)\in C^2$, $\psi(t)\in C^2$ $(0\leqq t<c)$.

(D) $\phi(t)\in C^2$, $\psi(t)\in C$ $(0\leqq t<c)$.

(E) $\phi(0)=\phi'(0)=\psi(0)=0$, $\phi''(0)>0$.

(F) $k=o(h)$ as $h, k\to+\infty$.

We propose to investigate how the following integrals approximate J when $h, k\to+\infty$, $a\to 0+$:

$$I_{a,h,k}=\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)t^2+k\psi'(0)t\right]dt, \tag{2}$$

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[3] Numbers in brackets refer to references at the end of the paper.

$$I'_{a,h,k} = \exp\,[-h\phi(\tau) + k\psi(\tau)] \cdot \int_0^a \exp\left[-\frac{1}{2}\,h\phi''(0)(t-\tau)^2\right] dt \tag{3}$$

where $\tau = (k/h)((\psi'(0)/\phi''(0)) + O(k^2/h^2)$. To obtain τ one solves the equation $h\phi'(\tau) = k\psi'(\tau)$ using conditions (B) and (F); see reference [2]. It should be noted here that $I_{a,h,k}$, $I'_{a,h,k}$ may conveniently be expressed as error functions.

We now prove the following theorems:

THEOREM 1. *If either* $\psi'(0) < 0$ *or* $k = O(h^{1/2})$, *and conditions* (A), (D), (E), (F) *are satisfied, then as* h, $k \to +\infty$, $a \to 0+$:

$$J \sim I_{a,h,k}.$$

THEOREM 2. *If* $\psi'(0) = 0$, *and conditions* (A), (C), (E), (F) *are satisfied, then as* h, $k \to +\infty$, $a \to 0+$:

$$J \sim I_{a,h,k}.$$

THEOREM 3. *If* $\psi'(0) > 0$, $h^{1/2} = o(k)$, *and conditions* (A), (B), (E), (F) *are satisfied, then as* h, $k \to +\infty$, $a \to 0+$:

$$J \sim I'_{a,h,k}$$

provided $(a - \tau)h^{1/2}$ *is bounded from below.*

The statement and proof of Theorem 4 covering the somewhat more complicated general case of $a < \tau$ appear after the proofs of Theorems 1–3.

To prove Theorem 1 we first choose $\epsilon > 0$ and for convenience assume $\phi''(0) > \epsilon$, $|\psi'(0)| > \epsilon$. Then we may determine an η such that $|\phi''(0) - \phi''(t)| < \epsilon$, $|\psi'(0) - \psi'(t)| < \epsilon$ for $0 < t < \eta \leqq c$. We then choose $a \leqq \eta$ since $a = o(1)$ by hypothesis. We then have

$$\left|(J - I_{a,h,k})/I_{a,h,k}\right|$$

$$= \left| \frac{\int_0^a \exp\,[-h\phi(t) + k\psi(t)]dt - \int_0^a \exp\left[-\frac{1}{2}\,h\phi''(0)t^2 + k\psi'(0)t\right] dt}{I_{a,h,k}} \right|$$

$$= \left| \frac{\int_0^a \left\{\exp\left[-\frac{1}{2}\,h\phi''(\theta_1 t)t^2 + k\psi'(\theta_2 t)t\right] - \exp\left[-\frac{1}{2}\,h\phi''(0)t^2 + k\psi'(0)t\right]\right\} dt}{I_{a,h,k}} \right| \quad (0 < \theta_1,\ \theta_2 < 1)$$

$$= \left| \frac{\int_0^a \exp\left[-\frac{1}{2}\,h\phi''(0)t^2 + k\psi'(0)t\right] \left\{\exp\left[-\frac{1}{2}\,h(\phi''(\theta_1 t) - \phi''(0))t^2 + k(\psi'(\theta_2 t) - \psi'(0))t\right] - 1\right\} dt}{I_{a,h,k}} \right|$$

$$\leqq \left| \frac{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)t^2+k\psi'(0)t\right]\left\{\exp\left[\frac{1}{2}h\epsilon t^2+k\epsilon t\right]-1\right\}dt}{I_{a,h,k}} \right|$$

$$\leqq \left| \frac{\epsilon\int_0^a \left\{\exp\left[-\frac{1}{2}h\phi''(0)t^2+k\psi'(0)t\right]\exp\left[\frac{1}{2}h\epsilon t^2+k\epsilon t\right]\right\}\{ht^2+kt\}dt}{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)t^2+k\psi'(0)t\right]dt} \right|$$

$$(4) = \left| \frac{\epsilon\int_0^{ah^{1/2}} \exp\left[-\frac{1}{2}(\phi''(0)-\epsilon)t^2+\frac{k}{h^{1/2}}(\psi'(0)+\epsilon)t\right]\left\{t^2+\frac{k}{h^{1/2}}t\right\}dt}{\int_0^{ah^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)t^2+\frac{k}{h^{1/2}}\psi'(0)t\right]dt} \right|$$

or

$$(5) \quad = \left| \frac{\epsilon\int_0^{ak} \exp\left[-\frac{1}{2}\frac{h}{k^2}(\phi''(0)-\epsilon)t^2+(\psi'(0)+\epsilon)t\right]\left\{\frac{h}{k^2}t^2+t\right\}dt}{\int_0^{ak} \exp\left[-\frac{1}{2}\frac{h}{k^2}\phi''(0)t^2+\psi'(0)t\right]dt} \right|.$$

First let us assume $ah^{1/2}$, ak are bounded away from zero (i.e. $ah^{1/2} \neq o(1)$, $ak \neq o(1)$). Then we have $|(J-I_{a,h,k})/I_{a,h,k}| = o(1)$ by (4) if $k = O(h^{1/2})$, and here no restriction is imposed on the sign of $\psi'(0)$. If $\psi'(0) < 0$, the one possibility not taken care of by (4) is when $h^{1/2} = o(k)$; in this case we have $|(J-I_{a,h,k})/I_{a,h,k}| = o(1)$ by (5). If we have $ah^{1/2} = o(1)$ and/or $ak = o(1)$, (4) and (5) may be rewritten as follows by using the mean value theorem for integrals:

$$(4.1) \quad = \left| \frac{\epsilon \exp\left[-\frac{1}{2}(\phi''(0)-\epsilon)t_1^2+\frac{k}{h^{1/2}}(\psi'(0)+\epsilon)t_1\right]\left\{t_1^2+\frac{k}{h^{1/2}}t_1\right\}}{\exp\left[-\frac{1}{2}\phi''(0)t_2^2+\frac{k}{h^{1/2}}\psi'(0)t_2\right]} \right|$$

$$(0 < t_1,\ t_2 < ah^{1/2}),$$

or

$$(5.1) \quad = \left| \frac{\epsilon \exp\left[-\frac{1}{2}\frac{h}{k^2}(\phi''(0)-\epsilon)t_3^2+(\psi'(0)+\epsilon)t_3\right]\left\{\frac{h}{k^2}t_3^2+t_3\right\}}{\exp\left[-\frac{1}{2}\frac{h}{k^2}\phi''(0)t_4^2+\psi'(0)t_4\right]} \right|$$

$$(0 < t_3,\ t_4 < ak),$$

and again we have $|(J-I_{a,h,k})/I_{a,h,k}| = o(1)$ for each of the cases above. This establishes Theorem 1.

To prove Theorem 2 we proceed in a similar manner. We choose $\epsilon>0$ as before and determine an η such that $|\phi''(0)-\phi''(t)|<\epsilon$ for $0<t<\eta\leqq c$. We now choose $a\leqq\eta$. We have $|\psi''(t)|<M_1$ in $0\leqq t<c$ since $\psi(t)\in C^2$ on $0\leqq t<c$. Hence we have

$$
\begin{aligned}
&|(J-I_{a,h,k})/I_{a,h,k}| \\
&= \left| \frac{\int_0^a \exp\left[-h\phi(t)+k\psi(t)\right]dt - \int_0^a \exp\left[-\frac{1}{2}h\phi''(0)t^2\right]dt}{I_{a,h,k}} \right| \\
&= \left| \frac{\int_0^a \left\{\exp\left[-\frac{1}{2}h\phi''(\theta_1 t)t^2+\frac{1}{2}k\psi''(\theta_1 t)t^2\right] - \exp\left[-\frac{1}{2}h\phi''(0)t^2\right]\right\}dt}{I_{a,h,k}} \right| \qquad (0<\theta_1<1) \\
&\leqq \left| \frac{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)t^2\right]\left\{\exp\left[\frac{1}{2}\epsilon ht^2+\frac{1}{2}kM_1t^2\right]-1\right\}dt}{I_{a,h,k}} \right| \\
&< \left| \frac{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)t^2\right]\left\{\exp\left[\frac{1}{2}(\epsilon h+kM_1)t^2\right]\right\}(\epsilon h+kM_1)t^2dt}{I_{a,h,k}} \right| \\
&= \left| \frac{\int_0^{ah^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)t^2+\frac{1}{2}\left(\epsilon+\frac{k}{h}M_1\right)t^2\right]\left\{\epsilon+\frac{k}{h}M_1\right\}t^2dt}{\int_0^{ah^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)t^2\right]dt} \right| .
\end{aligned}
$$

The quantity $[\phi''(0)-(\epsilon+(k/h)M_1)]$ can be made positive. Thus we have $J\sim I_{a,h,k}$ unless $ah^{1/2}=o(1)$, and this exceptional case may be handled with a mean value theorem for integrals just as in Theorem 1.

In proving Theorem 3 we choose $\epsilon>0$ as in proving Theorem 2 above so that we have $|\phi''(0)-\phi''(t)|<\epsilon$ for $0\leqq t<\eta\leqq c$; we again take $a\leqq\eta$. We have $|\psi''(t)|<M_1$ in $0\leqq t<c$ since $\psi(t)\in C^2$ in $0\leqq t<c$.

$$
\begin{aligned}
&|(J-I'_{a,h,k})/I'_{a,h,k}| \\
&= \left| \frac{\int_0^a \exp\left[-h\phi(t)+k\psi(t)\right]dt - \exp\left[-h\phi(\tau)+k\psi(\tau)\right]\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)(t-\tau)^2\right]dt}{I'_{a,h,k}} \right| \\
&= \left| \frac{\int_0^a \exp\left[(-h\phi''(t_1)+k\psi''(t_1))\frac{(t-\tau)^2}{2}\right]dt - \int_0^a \exp\left[-\frac{1}{2}h\phi''(0)(t-\tau)^2\right]dt}{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)(t-\tau)^2\right]dt} \right| \qquad (t_1 \text{ between } t \text{ and } \tau) \\
&= \left| \frac{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)(t-\tau)^2\right]\left\{\exp\left[-\frac{1}{2}h(\phi''(t_1)-\phi''(0))(t-\tau)^2+\frac{1}{2}k\psi''(t_1)(t-\tau)^2\right]-1\right\}dt}{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)(t-\tau)^2\right]dt} \right|
\end{aligned}
$$

$$\leqq \left| \frac{\int_0^a \exp\left[-\frac{1}{2} h\phi''(0)(t-\tau)^2\right]\left\{\exp\left[\frac{1}{2}(h\epsilon+kM_1)(t-\tau)^2\right]-1\right\}dt}{\int_0^a \exp\left[-\frac{1}{2}h\phi''(0)(t-\tau)^2\right]dt} \right|$$

$$= \left| \frac{\int_{-\tau}^{a-\tau} \exp\left[-\frac{1}{2} h\phi''(0)u^2\right]\left\{\exp\left[\frac{1}{2}(h\epsilon+kM_1)u^2\right]-1\right\}du}{\int_{-\tau}^{a-\tau} \exp\left[-\frac{1}{2}h\phi''(0)u^2\right]du} \right|$$

$$\leqq \left| \frac{\int_{-\tau h^{1/2}}^{(a-\tau)h^{1/2}} \exp\left[-\frac{1}{2} \phi''(0)t^2\right]\left\{\exp\left[\frac{1}{2}\left(\epsilon+\frac{k}{h}M_1\right)t^2\right]\right\}\left(\epsilon+\frac{k}{h}M_1\right)t^2dt}{\int_{-\tau h^{1/2}}^{(a-\tau)h^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)t^2\right]dt} \right|$$

$$(6) = \left| \frac{\left(\epsilon+\frac{k}{h}M_1\right)\int_{-\tau h^{1/2}}^{(a-\tau)h^{1/2}} \exp\left[\frac{1}{2}\left(-\phi''(0)+\epsilon+\frac{k}{h}M_1\right)t^2\right]t^2dt}{\int_{-\tau h^{1/2}}^{(a-\tau)h^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)t^2\right]dt} \right|.$$

The quantity $[\phi''(0)-(\epsilon+(k/h)M')]$ can again be made positive; both integrals in (6) are finite since $1/(\tau h^{1/2})=o(1)$, and the upper limit in each integral is bounded from below by hypothesis. Hence we have $J\sim I'_{a,h,k}$. However if $(a-\tau)h^{1/2}\to-\infty$, (6) is indeterminate, and we must consider another approach as is given below in Theorem 4.

We consider now in general the case when $a<\tau$; we have a approaching zero faster than τ. The main contribution to J will come from the neighborhood of $t=a$ and not necessarily from the neighborhood of $t=\tau$ as in Theorem 3. Therefore we must consider the expansion of $-h\phi(t)+k\psi(t)$ in powers of $(t-a)$. Also we must have

$$-h\phi'(a)+k\psi'(a)>0$$

since we have a maximum at $t=\tau$ for sufficiently large h, k. We adopt the notation:

$$\beta = -h\phi'(a) + k\psi'(a),$$

$$(7)\quad \begin{aligned} I''_{a,h,k} &= \exp\left[-h\phi(a)+k\psi(a)\right] \\ &\quad\cdot\int_0^a \exp\left\{[-h\phi'(a)+k\psi'(a)](t-a) - \frac{1}{2}h\phi''(0)(t-a)^2\right\}dt \\ &= \exp\left[-h\phi(a)+k\psi(a)\right] \\ &\quad\cdot\int_0^a \exp\left[\beta(t-a)-\frac{1}{2}h\phi''(0)(t-a)^2\right]dt. \end{aligned}$$

It should be noted that $I''_{a,h,k}$ may also be expressed as an error function. The behavior of J now depends on the quantity $\beta/h^{1/2}$, which is more complex than the ratio $k/h^{1/2}$ appearing in Theorems 1–3. We state the following theorem.

THEOREM 4. *If* $\psi'(0)>0$, $a<\tau$, *and conditions* (A), (B), (E), (F) *are satisfied, then* $J\sim I''_{a,h,k}$ *as* $h,k\to+\infty$, $a\to 0+$.

To prove Theorem 4 we choose $\epsilon>0$ so that we have $|\phi''(0)-\phi''(t)| <\epsilon$ for $0\leqq t<\eta\leqq c$; again we have $|\psi''(t)|<M_1$ in $0\leqq t<c$ since $\psi(t)\in C^2$ in $0\leqq t<c$. We have then, on taking $a\leqq\eta$ as before,

$$\begin{aligned}
&|(J-I''_{a,h,k})/I''_{a,h,k}|\\
&=\Bigg|\Big\{\exp\,[-h\phi(a)+k\psi(a)]\int_0^a\exp\left[\beta(t-a)+(-h\phi''(a_1)+k\psi''(a_1))\frac{(t-a)^2}{2}\right]dt\\
&\qquad -\exp\,[-h\phi(a)+k\psi(a)]\int_0^a\exp\left[\beta(t-a)-h\phi''(0)\frac{(t-a)^2}{2}\right]dt\Big\}\Big/I''_{a,h,k}\Bigg|
\end{aligned}$$

(a_1 between t and a)

$$=\left|\frac{\displaystyle\int_0^a\exp\left[\beta(t-a)-h\phi''(0)\frac{(t-a)^2}{2}\right]\left\{\exp\left(-h(\phi''(a_1)-\phi''(0))\frac{(t-a)^2}{2}+k\psi''(a_1)\frac{(t-a)^2}{2}\right)-1\right\}dt}{\displaystyle\int_0^a\exp\left[\beta(t-a)-\frac{1}{2}h\phi''(0)(t-a)^2\right]dt}\right|$$

$$\leqq\left|\frac{\displaystyle\int_0^a\exp\left[\beta(t-a)-h\phi''(0)\frac{(t-a)^2}{2}\right]\left\{\exp\left[(\epsilon h+kM_1)\frac{(t-a)^2}{2}\right]-1\right\}dt}{\displaystyle\int_0^a\exp\left[\beta(t-a)-\frac{1}{2}h\phi''(0)(t-a)^2\right]dt}\right|$$

$$=\left|\frac{\displaystyle\int_0^a\exp\,[-\beta u-h\phi''(0)u^2/2]\{\exp\,[(\epsilon h+kM_1)u^2/2]-1\}du}{\displaystyle\int_0^a\exp\,[-\beta u-h\phi''(0)u^2/2]du}\right|$$

$$\leqq\left|\frac{\displaystyle\int_0^a\exp\,[-\beta u-h\phi''(0)u^2/2]\exp\,[(\epsilon h+kM_1)u^2/2](\epsilon h+kM_1)u^2du}{\displaystyle\int_0^a\exp\,[-\beta u-h\phi''(0)u^2/2]du}\right|$$

$$(8)=\left|\frac{\displaystyle\left(\epsilon+\frac{k}{h}M_1\right)\int_0^{ah^{1/2}}\exp\left[-\frac{\beta}{h^{1/2}}u-\frac{1}{2}\phi''(0)u^2\right]\exp\left[\left(\epsilon+\frac{k}{h}M_1\right)\frac{u^2}{2}\right]u^2du}{\displaystyle\int_0^{ah^{1/2}}\exp\left[-\frac{\beta}{h^{1/2}}u-\frac{1}{2}\phi''(0)u^2\right]du}\right|$$

or

$$(9)=\left|\frac{\displaystyle\left(\epsilon+\frac{k}{h}M_1\right)\int_0^{a\beta}\exp\left[-u-\frac{1}{2}\frac{h}{\beta^2}\phi''(0)u^2\right]\exp\left[\left(\epsilon+\frac{k}{h}M_1\right)\frac{h}{\beta^2}u^2\right]\frac{h}{\beta^2}u^2du}{\displaystyle\int_0^{a\beta}\exp\left[-u-\frac{1}{2}\phi''(0)\frac{h}{\beta^2}u^2\right]du}\right|.$$

We note that $\epsilon+(k/h)M_1=o(1)$ since ϵ may be chosen arbitrarily small and $k=o(h)$ by hypothesis. Assuming first $ah^{1/2}\neq o(1)$, $a\beta\neq o(1)$ we have $J\sim I''_{a,h,k}$ by (8) if $\beta=O(h^{1/2})$ and by (9) if $h^{1/2}=o(\beta)$. If $ah^{1/2}=o(1)$ and/or $a\beta=o(1)$, we may use a mean value theorem to complete the proof as in Theorem 1.

It is of interest to evaluate some special cases of the integrals $I_{a,h,k}$, $I'_{a,h,k}$, $I''_{a,h,k}$. For example if $\psi'(0)<0$ and $h^{1/2}=o(k)$ we have from Theorem 1

$$J \sim I_{a,h,k} \sim \frac{[1 - \exp(\psi'(0)ak)]}{-k\psi'(0)}. \tag{10}$$

If $ak\to+\infty$, (10) would reduce to the result in Theorem 3 of [2]. It will readily be seen that to give more precise results more information must be known about the behavior of ak and $ah^{1/2}$, concerning which nothing is assumed in the present discussion. It may be shown from Theorem 3 in this paper that for $a=\tau$, $\psi'(0)>0$, and $h^{1/2}=o(k)$,

$$J \sim I'_{a,h,k} \sim \exp\left[-h\phi(a) + k\psi(a)\right]\left(\frac{\pi}{2\phi''(0)h}\right)^{1/2}. \tag{11}$$

From Theorem 4 we have for $\psi'(0)>0$, $a<\tau$, and $h^{1/2}=o(-h\phi'(a)+k\psi'(a))$,

$$J\sim I''_{a,h,k}\sim\frac{\exp\left[-h\phi(a)+k\psi(a)\right]\left\{1-\exp\left[ah\phi'(a)-ak\psi'(a)\right]\right\}}{-h\phi'(a)+k\psi'(a)}. \tag{12}$$

When ak, $ah^{1/2}\to+\infty$, the results will frequently be the same as in [2]; when this is not the case, most of the results explicitly involving a will be expressible as error functions with the exception of a few similar to those above.

References

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