ON THE EXTENSIONS OF BERNSTEIN POLYNOMIALS TO THE INFINITE INTERVAL[1]

P. L. BUTZER

1. **Introduction.** If the function $f(x)$ is defined in the infinite interval $0 \leqq x < \infty$, M. Kac,[2] J. Favard [**4**], and also O. Szász [**8**] considered the transform

$$P_u^f(x) = e^{-ux} \sum_{\nu=0}^{\infty} f(\nu/u) \frac{(ux)^\nu}{\nu!}, \qquad u > 0, \tag{1}$$

which can be used to approximate $f(x)$ as $u \to \infty$. This transform is the analogue for the interval $[0, 1]$ of the Bernstein polynomials

$$B_n^f(t) = \sum_{\nu=0}^{n} f(\nu/n) \binom{n}{\nu} t^\nu (1-t)^{n-\nu}, \qquad n = 1, 2, 3, \cdots .$$

Now if $f(x)$ is continuous in $(0, \infty)$ it is known (see [**8**] or [**4**]) that $\lim_{u\to\infty} P_u^f(x) = f(x)$ uniformly in $(0, \infty)$; for the interval $[0, 1]$ the corresponding familiar result holds for the Bernstein polynomials (see [**4**] or [**1**] for a recent approach to the *general* approximation problem in $[0, 1]$).

The purpose of this note is to study the approximation problem for a generalization of the transform $P_u^f(x)$ in the case $f(x)$ is an integrable function.

Bearing in mind the definition of uniform convergence of a sequence of functions at a point (cf. [**9**, p. 26]), Szász has proved the following.

LEMMA 1. *If the rth derivative $f^{(r)}(x)$ exists and $f^{(r)}(x) = O(x^k)$ as $x \to \infty$, for some $k > 0$, and if $f^{(r)}(x)$ is continuous at the point $x = \zeta$, then $P_u^{(r)}(x)$ approaches $f^{(r)}(x)$ uniformly at $x = \zeta$.*

Also we need two other lemmas.

LEMMA 2.

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[2] Professor M. Kac very kindly pointed out to me that he considered the transform (1) several years ago and made use of it in his lectures and seminars, but never published his results.

$$(2) \qquad \sum_{\nu=0}^{\infty} (\nu - u)^2 \frac{u^\nu}{\nu!} = ue^u.$$

LEMMA 3. *For* $0<\delta<1$

$$\sum_{|\nu-u|>\delta u} e^{-u} \frac{u^\nu}{\nu!} = O\left\{\exp\left(-\frac{\delta^2 u}{3}\right)\right\}, \qquad u \to \infty.$$

The second lemma is readily verified and for the third see [**5**, p. 200].

2. Approximations by the derivatives of the transform $P_u^f(x)$.

THEOREM 1. *If $f(x)$ is bounded in the interval $0 \leqq x \leqq R$ for every $R>0$, $f(x)=O(x^k)$, $x\to\infty$, for some $k>0$, then at every point ζ where $f'(\zeta)$ exists,*

$$(3) \qquad \lim_{u\to\infty} \frac{d}{d\zeta} P_u(\zeta) = \frac{d}{d\zeta} f(\zeta).$$

PROOF. We have

$$(4) \qquad \frac{d}{d\zeta} P_u(\zeta) = ue^{-u\zeta} \sum_{\nu=0}^{\infty} [f(\nu + 1/u) - f(\nu/u)] \frac{(u\zeta)^\nu}{\nu!}$$

and since $u[f(x+1/u)-f(x)]\to f'(x)$ if the derivative exists, the relation (3) holds at the point $\zeta=0$, if $f'(0)$ exists.

Let $\zeta>0$. We may write $f(x)=f(\zeta)+(x-\zeta)f'(\zeta)+\epsilon(x)(x-\zeta)$ where $|\epsilon(x)| \leqq \eta(\delta)$ for $|x-\zeta| \leqq \delta$ and $\eta(\delta)\to 0$ for $\delta\to 0$. Let $p(x)$ represent the polynomial $f(\zeta)+(x-\zeta)f'(\zeta)$ and let $g(x)=(x-\zeta)\epsilon(x)$ and therefore $g(\zeta)=g'(\zeta)=0$. Hence

$$\frac{d}{dx} P_u^f(x) = \frac{d}{dx} P_u^p(x) + \frac{d}{dx} P_u^g(x)$$

and, by Lemma 1, to prove our theorem we only need to show that the second term on the right-hand side approaches zero as $x\to\zeta$. It follows readily that

$$\frac{d}{d\zeta} P_u^g(\zeta) = \frac{e^{-u\zeta}}{u\zeta} \sum_{\nu=0}^{\infty} \epsilon_\nu(u)(\nu - u\zeta)^2 \frac{(u\zeta)^\nu}{\nu!}$$

where $|\epsilon_\nu(u)| \leqq \eta(\delta)$ for $|\nu/u-\zeta| \leqq \delta$. We write

$$\sum_{\nu=0}^{\infty} \epsilon_\nu(u)(\nu - u\zeta)^2 \frac{(u\zeta)^\nu}{\nu!} = \sum_{|\nu-u\zeta|\leqq u\delta} + \sum_{u\zeta-\nu>u\delta} + \sum_{\nu-u\zeta>u\delta}$$
$$= T_1 + T_2 + T_3 \quad \text{say.}$$

By Lemma 2,

$$\frac{e^{-u\zeta}}{u\zeta}\left| T_1 \right| < \eta(\delta) \frac{e^{-u\zeta}}{u\zeta}\cdot u\zeta e^{u\zeta} = \eta(\delta).$$

Denoting $\sup_{x\leqq\zeta}|f(x)| = M(\zeta)$ and noting that

$$\epsilon_\nu(u)(\nu - u\zeta)^2 = u(\nu - u\zeta)[f(\nu/u) - f(\zeta)] - (\nu - u\zeta)^2 f'(\zeta),$$

we obtain

$$\left| T_2 \right| < [\zeta M(\zeta) + \zeta^2 |f'(\zeta)|] \sum_{u\zeta-\nu>u\delta} u^2 \frac{(u\zeta)^\nu}{\nu!} \cdot$$

From Lemma 3, we deduce

$$\frac{e^{-u\zeta}}{u\zeta} T_2 = O\left\{u \exp\left(-\frac{\delta^2 u}{3\zeta}\right)\right\}.$$

Since $f(x) = O(x^k)$, for $\nu > u\zeta$ and $k \geqq 1$, we have

$$\epsilon_\nu(u)(\nu - u\zeta)^2 = Ou^2\left(\frac{\nu^{k+1}}{u^{k+1}}\right),$$

therefore

$$T_3 = O\left(\sum_{\nu-u\zeta>u\delta} u^2 \frac{\nu^{k+1}}{u^{k+1}} \frac{(u\zeta)^\nu}{\nu!}\right) = O\left(\sum_{\nu-u\zeta>u\delta-k-1} u^2 \frac{(u\zeta)^\nu}{\nu!}\right)$$

$$= O\left\{u^2 \exp\left(u\zeta - \frac{\delta^2 u}{3\zeta}\right)\right\}$$

by Lemma 3. Hence

$$\frac{e^{-u\zeta}}{u\zeta} T_3 = O\left\{u \exp\left(-\frac{\delta^2 u}{3\zeta}\right)\right\}$$

and we finally obtain

$$\left| \frac{d}{d\zeta} P_u^g(\zeta) \right| \leqq \eta(\delta) + O\left\{u \exp\left(-\frac{\delta^2 u}{3\zeta}\right)\right\}$$

and since δ is arbitrarily small, our theorem is established.

This result can also be generalized to higher derivatives. For the analogous theorem in the case of Bernstein polynomials, see [6].

3. **A transform for integrable functions.** We now consider a generalization of the transform $P_u^f(x)$ for integrable functions.

Let $f(x)$ be Lebesgue integrable over the interval $0 \leqq x \leqq R$ for every $R > 0$ and $F(x) = O(x^k)$ as $x \to \infty$, for some $k > 0$, where $F(x) = \int_0^x f(s)ds$. We define

$$W_u^f(x) = ue^{-ux}\sum_{\nu=0}^{\infty}\left[\int_{\nu/u}^{(\nu+1)/u} f(s)ds\right]\frac{(ux)^\nu}{\nu!}, \qquad u > 0,$$

which can be written as

$$W_u^f(x) = \int_0^\infty K_u(x;\,s)f(s)ds$$

where

$$K_u(x;\,s) = ue^{-ux}\frac{(ux)^\nu}{\nu!} \tag{5}$$

for $\nu/u < s \leqq (\nu+1)/u$, $\nu = 0, 1, 2, \cdots$; $K_u(x;\,0) = 0$; $0 \leqq x < \infty$.

From (4) we have

$$W_u^f(x) = \frac{d}{dx}P_u^F(x)$$

and, employing Theorem 1, we obtain:

THEOREM 2. *If $f(x) \in L(0,\,\infty)$, then*

$$\lim_{u\to\infty} W_u^f(x) = f(x)$$

at every point where

$$f(x) = F'(x),$$

i.e. almost everywhere in $(0,\,\infty)$.

LEMMA 4. *Let $\chi_u(x;\,s)$, fixed $0 \leqq x < \infty$, be a non-negative function of bounded variation in the variable s, $0 < s < \infty$, depending on the parameter u, with $\chi_u(x;\,0) = 0$ and satisfying*

$$\int_0^\infty \chi_u(x;\,s)ds \leqq C, \tag{6}$$

$$\int_0^\infty |s - x|\,|d_s\chi_u(x;\,s)| \leqq C_1, \tag{7}$$

both integrals existing, where C, C_1 are independent of u. Define

$$H_u(x) \equiv \int_0^\infty \chi_u(x;\,s)f(s)ds$$

where $f \in L(0,\,\infty)$. Then

$$\sup_{u>0} |H_u(x)| \leqq (C + 2C_1)\theta(x; f)$$

where

$$\theta(x; f) = \sup_{0<s<\infty, s\neq x} \frac{1}{s - x} \int_x^s |f(y)| \, dy.$$

This lemma may be established by a modification of the proof for Lemma 2 of [2].

LEMMA 5. *For* $u > 0$, $0 \leqq x < \infty$,

$$\sum_{\nu=0}^{\infty} \left| \frac{\nu + 1}{u} - x \right| \left| ue^{-ux} \left(\frac{1}{\nu!} (ux)^\nu - \frac{1}{(\nu + 1)!} (ux)^{\nu+1} \right) \right| \leqq 1. \tag{8}$$

The left-hand side of (8) equals

$$ue^{-ux} \sum_{\nu=0}^{\infty} \left| \frac{\nu + 1}{u} - x \right| \left| \frac{(ux)^\nu}{\nu!} \left[1 - \frac{ux}{\nu + 1} \right] \right|$$

$$= \frac{e^{-ux}}{ux} \sum_{\nu=0}^{\infty} [(\nu + 1) - ux]^2 \frac{(ux)^{\nu+1}}{(\nu + 1)!} \leqq \frac{e^{-ux}}{ux} uxe^{ux} = 1$$

by Lemma 2. This proves Lemma 5.

Now our kernel (5) satisfies the hypothesis of Lemma 4. In fact the inequality (8) shows that (7) holds with $C_1 = 1$ and since

$$\int_0^\infty K_u(x; s) ds = e^{-ux} \sum_{\nu=0}^{\infty} \frac{(ux)^\nu}{\nu!} = 1$$

the relation (6) holds with $C = 1$.

If $f(x) \in L(0, \infty)$ we hence obtain that the $W_u^f(x)$, for every $u > 0$, have as a majorant the function $3\theta(x; f)$, in other words

$$\sup_{u>0} |W_u^f(x)| \leqq 3\theta(x; f). \tag{9}$$

We now come to our main theorem.

THEOREM 3. *If* $f(x)$ *and* $|f(x)|^p$, $p > 1$, *both belong to* $L(0, \infty)$, *the transform* $W_u^f(x)$ *converges dominatedly to* $f(x)$ *almost everywhere in the space* L^p; *in other words*

(i) $W_u^f(x) \to f(x)$ *p.p. in* $(0, \infty)$,

(ii) *there is an element* $\theta(x; f)$ *in the space* L^p *such that*

$$\sup_{u>0} |W_u^f(x)| \leqq 3\theta(x; f) \text{ p.p. } x \text{ in } (0, \infty).$$

For, if $f \in L^p(0, \infty)$, $p>1$, it is known that (see [9, p. 244]) $\theta(x; f) \in L^p(0, \infty)$ with

$$\int_0^\infty \theta^p(x; f)dx \leqq 2\left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p dx$$

and by (9) and Theorem 2, this theorem is immediate.

COROLLARY. *Under the hypothesis of the previous theorem*

$$\int_0^\infty |W_u^f(x) - f(x)|^p dx \to 0, \qquad u \to \infty.$$

For Bernstein polynomials, results analogous to the above have been given by the author [2].

It may be of interest to mention that one may also formulate the results of the previous theorem in terms of the Banach space $\Lambda(\phi, p)$ for the infinite interval $(0, \infty)$, introduced by Lorentz [7].

4. **Extensions of a different type.** Another representation of Bernstein polynomials corresponding to functions defined over an infinite interval has been given by Chlodovsky [3].

Let $b_1, b_2, \cdots$ be a sequence of positive numbers such that

$$b_1 < b_2 < b_3 < \cdots < b_n < \cdots, \qquad \lim_{n\to\infty} b_n = +\infty.$$

Let $f(x)$ be defined on $0 \leqq x < \infty$. Then Chlodovsky considered the polynomials

$$B_n^f(x; b_n) = \sum_{\nu=0}^{n} f\left(\frac{b_n \nu}{n}\right)\binom{n}{\nu}\left(\frac{x}{b_n}\right)^\nu\left(1 - \frac{x}{b_n}\right)^{n-\nu},$$

and studied their convergence to $f(x)$, see [3] or [6].

For these polynomials, corresponding to Theorem 1, one may establish the following result (which we only state).

THEOREM 4. *If* $|f(x)| \leqq M$ *on* $0 \leqq x < \infty$, $b_n = o(n^\beta)$, $n \to \infty$, $0 < \beta < 1/2$, *then*

$$B_n'(\zeta; b_n) \to f'(\zeta)$$

at every point ζ *where the latter derivative exists.*

A result similar to Theorem 3 can also be established for polynomials which are generalizations of the $B_n^f(x; b_n)$ for integrable functions.

Finally we would like to add that in case of the double infinite interval $(-\infty, +\infty)$, Favard [4] introduced the transform

$$\frac{1}{(u\pi)^{1/2}} \sum_{\nu=-\infty}^{+\infty} f(\nu/u) e^{-u(\nu/u-x)}$$

and considered its convergence to $f(x)$ under the assumption $f(x)$ is defined and continuous on $(-\infty, +\infty)$.

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McGILL UNIVERSITY