SOME IMPLICATIONS OF SEMI-1-CONNECTEDNESS[1]

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A topological space X is said to be semi-n-connected at a point $x \in X$ provided there exists an open set U containing x such that every Čech n-cycle on a compact subset of U bounds on X. (If X is regular, "a compact subset of" may be omitted.) If X is semi-n-connected at every point, then X is said to be semi-n-connected [1].[2] As shown in [1], if a space has a finite 1-dimensional Betti number, it is semi-1-connected, but the converse is not true. The purpose of this article is to prove certain "point set" properties implied by the hypothesis that the space is semi-1-connected.

In what follows it is assumed that X is a separable metric space. The coefficient group will be a field F.

THEOREM 1. *If a continuum X is semi-1-connected at $x \in X$, then there exists an open set V containing x such that, if Q is a region (connected open set) with $\overline{Q} \subset V$, then no continuum in $V-Q$ intersects two components of $F(Q)$. ($F(Q)$ shall mean $\overline{Q}-Q$ for any open set Q.)*

PROOF. Let V be an open set such that $x \in V$ and every Čech 1-cycle on V bounds on X. Suppose Q is a region with $\overline{Q} \subset V$ and that K is a continuum in $V-Q$ which intersects two components H_1 and H_2 of $F(Q)$. Let $x \in H_1 \cdot K$ and $y \in H_2 \cdot K$ and let U be an open set such that $K+\overline{Q} \subset U \subset \overline{U} \subset V$. The set $\{x\}+\{y\}$ generates a nonzero element w of the reduced zero dimensional group, $\tilde{H}_0(F(Q))$, since x and y belong to different components of $F(Q)$.

Consider the portion of the Mayer-Vietoris sequence for the triad $(\overline{U};\, \overline{U}-Q,\, \overline{Q})$:

$$\xrightarrow{f} H_1(\overline{U}) \xrightarrow{g} \tilde{H}_0(F(Q)) \xrightarrow{h} \tilde{H}_0(\overline{Q}) \oplus \tilde{H}_0(\overline{U}-Q).$$

The image of w under h is obtained as follows: if $i_1\colon F(Q) \to \overline{Q}$ and $i_2\colon F(Q) \to \overline{U}-Q$ are inclusion mappings, then $h(w) = (i_{1*}(w),\, i_{2*}(w)) \in \tilde{H}_0(\overline{Q}) \oplus \tilde{H}_0(\overline{U}-Q)$. Since $\overline{Q}$ is connected, $i_{1*}(w)=0$, and since both x and y are in the same component of $\overline{U}-Q$, $i_{2*}(w)=0$. Therefore, w

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

is in the kernel of h and, by exactness, there exists a $z \in H_1(\overline{U})$ such that $g(z) = w$.

Consider now the Mayer-Vietoris sequence for the triad $(X; X-Q, \overline{Q})$. If $j: (\overline{U}; \overline{U}-Q, \overline{Q}) \to (X; X-Q, \overline{Q})$ is inclusion, then j induces a homomorphism of the Mayer-Vietoris sequence of $(\overline{U}; \overline{U}-Q, \overline{Q})$ into that of $(X; X-Q, \overline{Q})$. By the definition of such a homomorphism, we have commutativity in the diagram below.

$$\begin{array}{ccccc} \cdots \to H_1(X) & \xrightarrow{g'} & \tilde{H}_0(F(Q)) & \to & \tilde{H}_0(\overline{Q}) \oplus \tilde{H}_0(X-Q) \\ \uparrow j_* & & \uparrow (j \mid F(Q))_* & & \\ \cdots \to H_1(\overline{U}) & \xrightarrow{g} & \tilde{H}_0(F(Q)) & \to \cdots & \end{array}$$

Since every Čech 1-cycle on $\overline{U}$ bounds on X, $j_*(z) = 0$; hence, $g'j_*(z) = 0$. But $g'j_*(z) = (j \mid F(Q))_* g(z) = w$, since $(j \mid F(Q))_*$ is the identity, and $w \neq 0$. This contradiction implies the theorem is true.

A space X is said to be locally peripherally connected at $x \in X$ provided that for every open set P containing x there exists an open set M containing x and contained in P such that $F(M)$ is connected. A point $x \in X$ is a local separating point if $V - \{x\}$ is not connected for some open connected set V containing x. The next theorem proves a relation between these concepts.

THEOREM 2. *If X is a locally connected continuum which is semi-1-connected at a nonlocal separating point $x \in X$, then X is locally peripherally connected at x.*

PROOF. Let U be an open set such that $x \in U$ and every Čech 1-cycle on U bounds on X. If X is not locally peripherally connected at x, then there exists an open set M containing x and contained in U such that no open set contained in M and containing x has a connected boundary. M may be taken to have property S [**2**, p. 22, §15.43], to be connected, and to be such that $\overline{M} \subset U$. For each positive integer i, let R_i be a region such that $x \in R_i$, diam $(R_i) \leq (1/(i+1))\rho(x, F(M))$, and $R_i \supset \overline{R}_{i+1}$.

Let $y \in M - R_1$. That every arc xy in U has its last point, in the order x to y, in $\overline{R}_i$, in the same component H_i of $F(R_i)$, for each i, is implied by Theorem 1. Hence, for each i, H_i separates x from y in U and, therefore, H_i separates x from y in $\overline{M}$. Let X_i be the component of $\overline{M} - H_i$ containing x. Since $H_{i+1} \subset R_i \subset X_i$, for each i, it is true that $X_{i+1} \subset X_i$ (see [**2**, p. 42]). The collection $\{X_i\}$ then forms a decreasing sequence: $X_1 \supset X_2 \supset X_3 \supset \cdots$.

Suppose that, for some i, $X_i \subset M$. Then, since X_i is open and since $F(X_i) = H_i$, which is connected, the selection of M is contradicted.

Hence, for each i, $X_i \cdot F(M) \neq 0$. Select now a sequence of points $\{x_i\}$ such that $x_i \in X_i$, as follows: (1) Choose x_1 from $X_1 \cdot F(M)$. (2) If $x_1 \in X_2$, then let $x_2 = x_1$; otherwise, let x_2 be any point in $X_2 \cdot F(M)$. (3) Supposing x_i has been chosen, choose $x_{i+1} = x_i$, if $x_i \in X_{i+1}$, or choose $x_{i+1} \in X_{i+1} \cdot F(M)$, if $x_i \notin X_{i+1}$. Suppose that the sequence resulting is infinite. Let $\bar{x}$ be a limit point of $\{x_i\}$, and let $\{x_{i_n}\}$ be a subsequence such that if $p<q$, then $i_p < i_q$ and $x_{i_p} \neq x_{i_q}$ and such that $x_{i_n} \to \bar{x}$ as $n \to \infty$. Since $F(M)$ is compact, $\bar{x} \in F(M)$, and, since $\overline{M}$ is locally connected [2, p. 20, §15.3], there exists a connected set N, open in $\overline{M}$, containing $\bar{x}$ and not intersecting $\overline{R}_1$. Let x_{i_p} and x_{i_q}, where $p<q$, be two points of $\{x_{i_n}\} \cdot N$. $N \subset \overline{M} - H_{i_q}$ and $x_{i_q} \in N$; therefore, $N \subset X_{i_q} \subset X_{i_q - 1} \subset \cdots \subset X_{i_p+1} \subset X_{i_p}$. Since $x_{i_p} \in N \subset X_{i_p+1}$, it is true that $x_{i_p+1} = x_{i_p}$, and since $x_{i_p+1} \in N \subset X_{i_p+2}$, $x_{i_p+2} = x_{i_p+1}$, etc. Thus, $x_{i_q} = x_{i_p}$, which is a contradiction.

Therefore all but a finite number of elements of the sequence $\{x_i\}$ are equal. Let I be a positive integer such that $i \geqq I$ and $j \geqq I$ imply $x_i = x_j$. Call this point $\bar{x}$. Since x is not a local separating point, $M - \{x\}$ is connected. Let, then, $\bar{x}y$ be an arc joining $\bar{x}$ to y in $\overline{M} - \{x\}$. There exists a positive integer J such that $i \geqq J$ implies $\overline{R}_i \cdot \bar{x}y = 0$. Let $\bar{x}x$ be an arc in X_{i_0} for $i_0 = \max(I, J)$. Let z_1 be the first point of $\bar{x}x \cdot F(R_{i_0})$, in the order $\bar{x}$ to x (then $z_1 \notin H_{i_0}$, obviously) and let z_2 be the last point of $xy \cdot F(R_{i_0})$, in the order x to y. Then $z_2 \in H_{i_0}$. The continuum $z_1\bar{x} + \bar{x}y + yz_2 \subset \overline{M} - R_{i_0} \subset U - R_{i_0}$ and intersects two different components of $F(R_{i_0})$, which contradicts the preceding theorem.

Therefore, X is locally peripherally connected at x.

THEOREM 3. *If X is a locally connected continuum which is semi-1-connected at a nonlocal separating point $x \in X$, then there exist arbitrarily small open sets V containing x such that, if Q is any region contained in V, then there exists a component C of $F(Q)$ such that the component of $X - C$ containing Q is contained in V.*

PROOF. Let U be an open set containing x and such that every Čech 1-cycle on U bounds on X. We may take U to be connected. By Theorem 2, there exist arbitrarily small open sets containing x and having connected boundaries. Let V be any one of these such that $\overline{V} \subset U$. Suppose Q is any open connected set contained in V. Let pv be any arc in U joining $p \in Q$ to $v \in F(V)$. Let C be the component of $F(Q)$ such that the last point z of $F(Q) \cdot pv$, in the order p to v, lies in C. Theorem 1 implies that every arc joining p to v has its last point in $F(Q)$, in the order p to v, in C. Since Q is arcwise connected and $F(V)$ is connected, the same statement may be made for any arc in U joining a point of Q to a point of $F(V)$.

Let Z be the component of $X-C$ containing Q. If Z is not contained in V, then $Z\cdot F(V)\neq 0$ and, Z being arcwise connected, there exists an arc pv in Z, connecting a point $p\in Q$ to $v\in Z\cdot F(V)$. Therefore, $pv\cdot C=0$. Let v' be the first point, in the order p to v, of $pv\cdot F(V)$. Then $pv'\subset \overline{V}\subset U$ and $pv'\cdot C\neq 0$. This contradiction implies $Z\subset V$.

THEOREM 4. *If T is a compact totally disconnected set in a locally connected continuum X such that no point of T is a local separating point of X and X is semi-1-connected at each $x\in T$, and if ϵ is a positive number, then there exists a finite open covering $\{Y_1, Y_2, \cdots, Y_l\}$ of T such that* (1) diam $(Y_i)<\epsilon$, *for each i,* (2) *$F(Y_i)$ is connected, for each i, and* (3) $Y_i\cdot Y_j=0$, *if $i\neq j$.*

PROOF. Let $\{U_x|x\in T\}$ be an open covering of T such that for each $x\in T$, every Čech 1-cycle on U_x bounds on X. By Theorem 2, there exists, for each $x\in T$, an open set V_x such that (1) $\overline{V}_x\subset U_x$, (2) $F(V_x)$ is connected, and (3) diam $(V_x)<\epsilon$. Let $\{V_1, V_2, \cdots, V_n\}$ be a finite subcover of the covering $\{V_x|x\in T\}$. Since T is compact, there exists a positive number d such that every subset Q of X with diam $(Q)<d$ and $Q\cdot T\neq 0$ is contained in V_j, for some $j\in\{1, \cdots, n\}$. Cover T with connected open sets $W_1, W_2, \cdots$ such that diam (W_i) $<d$, for each i, and $W_i\cdot W_j=0$, if $i\neq j$. Then, given any W_i, there exists a $j\in\{1, \cdots, n\}$ such that $\overline{W}_i\subset V_j$. Let $\{W_1, \cdots, W_p\}$ be a finite subcover (renumbered). Further, let $x_1, \cdots, x_n$ be points of T such that $U_{x_i}\supset \overline{V}_i$, for each $i\in\{1, \cdots, n\}$, and, for each $i\in\{1, \cdots, p\}$, let k_i be the smallest integer in $\{1, \cdots, n\}$ such that $V_{k_i}\supset \overline{W}_i$. Choose the component C_i of $F(W_i)$ as in the proof of Theorem 3 using a point $v_{k_i}\in F(V_{k_i})$ so that the component Z_i of $X-C_i$ containing W_i is contained in V_{k_i}. (Actually, $\overline{Z}_i\subset V_{k_i}$, here.) There exist at most p Z_i's.

(i) *If $i\neq j$, then either $Z_i\cdot Z_j=0$, or $Z_i\supset Z_j$, or $Z_j\supset Z_i$.*

PROOF. Suppose $Z_i\cdot Z_j\neq 0$. If $Z_i\cdot C_j=0$, then $Z_i\subset Z_j$, and if $Z_j\cdot C_i=0$, $Z_j\subset Z_i$. If neither is true, then both $Z_i\cdot C_j$ and $Z_j\cdot C_i$ are not empty. Since $W_i\subset X-C_j$ and has a boundary point in Z_j, which is open, $Z_j\cdot W_i\neq 0$ and $W_i\subset Z_j$. Similarly, $W_j\subset Z_i$. Let $p\in W_i$ and $q\in W_j$. Let pq be an arc in Z_i and let y be the last point of $(pq)\cdot F(W_i)$, in the order p to q. Then $y\in Z_i$, hence $y\notin C_i$. Let C_i' be the component of $F(W_i)$ containing y. Let pv_{k_i} be an arc in $U_{x_{k_i}}$ and let x be the last point of $(pv_{k_i})\cdot F(W_i)$, in the order p to v_{k_i}. Then $x\in C_i$.

Let qv_{k_i} be an arc in $U_{x_{k_i}}$ and let the last point of qv_{k_i} in $F(W_j)$, in the order q to v_{k_i}, be z. If $(v_{k_i}z)\cdot W_i=0$, then $K=xv_{k_i}+v_{k_i}z+\overline{W}_j+qy$ is a continuum in $U_{x_{k_i}}$ containing points in two different components

C_i and C_i' of $F(W_i)$, contradicting Theorem 1. If $(v_{k_i}z)\cdot W_i\neq 0$, let w be the last point of $(v_{k_i}z)\cdot F(W_i)$, in the order v_{k_i} to z. If $w\notin C_i'$, then $wz+\overline{W}_j+qy$ provides the same contradiction as above.

If $w\in C_i'$, then consider two cases:

Case 1, $k_i=k_j$. In this case observe that $z\in C_j$ and, since both p and q belong to Z_j, there exists an arc $(pq)'$ in Z_j. Let t be the last point of $(pq)'\cdot F(W_j)$, in the order q to p. Let C_j' be the component of $F(W_j)$ containing t. Since $t\in Z_j$ and $Z_j\cdot C_j=0$, $C_j'\neq C_j$. Then $zw\subset U_{x_{k_i}}-W_j$, $tp\subset U_{x_{k_i}}-W_j$, and $\overline{W}_i\subset U_{x_{k_i}}-W_j$. Therefore, $zw+\overline{W}_i+tp$ is a continuum in $U_{x_{k_i}}-W_j$ containing points in two components C_j and C_j' of $F(W_j)$, contradicting Theorem 1.

Case 2, $k_i\neq k_j$ (suppose $k_i<k_j$). In this case, we have $W_j\subset V_{k_i}$ but $\overline{W}_j\not\subset V_{k_i}$ (for, if it were, then in the selection of C_j the point v_{k_i} would have been used; that is, then $k_i=k_j$). Hence $\overline{W}_j\cdot F(V_{k_i})\neq 0$. Let xv_{k_i} and pq be as above. Then $xv_{k_i}\subset U_{x_{k_i}}-W_i$, $F(V_{k_i})\subset U_{x_{k_i}}-W_i$, $\overline{W}_j\subset U_{x_{k_i}}-W_i$, and $qy\subset U_{x_{k_i}}-W_i$. The continuum $xv_{k_i}+F(V_{k_i})+\overline{W}_j+qy$ is contained in $U_{x_{k_i}}-W_i$ and intersects two different components, C_i and C_i', of $F(W_i)$, contradicting Theorem 1.

Therefore, if $i\neq j$, then either $Z_i\cdot Z_j=0$, or $Z_i\subset Z_j$, or $Z_j\subset Z_i$.

Now let Y_1 be the sum of all Z_i such that $Z_i\cdot Z_1\neq 0$. Let k_2 be the smallest integer such that $Z_{k_2}\cdot Y_1=0$. Let Y_2 be the sum of all Z_i such that $Z_i\cdot Z_{k_2}\neq 0$. Continuing in this way, if Y_i has been defined, let k_{i+1} be the smallest integer such that $Z_{k_{i+1}}\cdot(Y_1+Y_2+\cdots+Y_i)=0$, and let Y_{i+1} be the sum of all Z_j such that $Z_j\cdot Z_{k_{i+1}}\neq 0$. Let $\{Y_1, Y_2,\cdots, Y_l\}$ be the collection thus obtained. Each Y_i, $i\in\{1,\cdots,l\}$, is a connected open set.

(ii) *If* $i\neq j$, *then* $Y_i\cdot Y_j=0$.

PROOF. Suppose $i<j$. If $Y_i\cdot Y_j\neq 0$, let $p\in Y_i\cdot Y_j$. Then there exists a Z_k such that $p\in Z_k$ and $Z_k\cdot Z_{k_i}\neq 0$, and a Z_m such that $p\in Z_m$ and $Z_m\cdot Z_{k_j}\neq 0$. Therefore $Z_k\cdot Z_m\neq 0$ and either (1) $Z_k\subset Z_m$ or (2) $Z_m\subset Z_k$. In case (1), $Z_m\cdot Z_{k_i}\neq 0$, so $Z_m\subset Y_i\subset\sum_{\alpha=1}^{\alpha=j-1} Y_\alpha$. But then $Z_{k_j}\cdot\sum_{\alpha=1}^{\alpha=j-1} Y_\alpha\neq 0$, which is a contradiction. In case (2), $Z_m\subset Y_i$ also, since $Z_m\subset Z_k$ and $Z_k\subset Y_i$, giving the same contradiction.

(iii) *For each* i, *there exists a* k *such that* $Y_i=Z_k$ *and a* j *such that* $Y_i\subset V_j$.

PROOF. Observe that there is a k such that $Z_k=Y_i$, for each i, for if $Z_{k_i}\neq Y_i$, then there is a point $p\in Y_i-Z_{k_i}$ and there is a j_1 such that $Z_{j_1}\cdot Z_{k_i}\neq 0$ and $p\in Z_{j_1}$. Therefore $Z_{j_1}\supset Z_{k_i}$ properly. If $Z_{j_1}\neq Y_i$, then there is a j_2 such that $Z_{j_2}\supset Z_{j_1}$ properly. But this process must stop since there are only a finite number of the Z's. Hence, there is a k such that $Z_k=Y_i$, for each i. That there is a j such that $Y_i\subset V_j$ is obvious.

The collection $\{Y_1, \cdots, Y_l\}$ satisfies the conclusion of the theorem, since

(1) For each i, there exists a j such that $Y_i \subset V_j$, so that diam $(Y_i) \leqq$ diam $(V_j) < \epsilon$.

(2) For each i, there exists a k such that $Y_i = Z_k$, so that $F(Y_i) = F(Z_k) = C_k$ is connected.

(3) $Y_i \cdot Y_j = 0$, if $i \neq j$.

Bibliography

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