ON THE PROOF OF THE MAIN TAUBERIAN THEOREM FOR THE C_k- AND H_k-METHODS[1]

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There are many proofs of the now classical one-sided Tauberian theorem for the C_k-method:

THEOREM I. *If the* a_ν *are real, if* $k>0$ *and*[2]

$$\sum a_\nu = s\ (C_k), \tag{1}$$

$$\nu a_\nu \leqq M \qquad (\nu = 0, 1, 2, \cdots), \tag{2}$$

M being independent of ν, *then*

$$\sum a_\nu = s. \tag{3}$$

The proofs are, for general k, not quite easy. The following inductive proof, taking the theorem for granted for $k=1$—this is the fundamental Hardy-Landau theorem[3] from 1910—seems to me of simpler structure than all of them, its main difference from these lying in the fact that it uses the series-to-series instead of the sequence-to-sequence transform.

We prove it first for the H_k-method, both theorems being identical for $k=1$:

THEOREM II. *If the* a_ν *are real, if* k *is an integer* $\geqq 2$, *and if*

$$\sum a_\nu = s\ (H_k) \tag{1'}$$

and (2) *are valid, then also* (3).

For if we put $a_\nu^{(0)} = a_\nu$ $(\nu=0, 1, \cdots)$, and $a_0^{(p+1)} = a_0$ for $p = 0, 1, \cdots$, and

$$a_\nu^{(p+1)} = \frac{a_1^{(p)} + 2a_2^{(p)} + \cdots + \nu a_\nu^{(p)}}{\nu(\nu+1)} \qquad (\nu = 1, 2, \cdots), \tag{4}$$

then (1′) is the same as

$$\sum a_\nu^{(p+1)} = s \qquad \text{for } p = k-1 \tag{5}$$

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[2] The sums $\sum$ are always taken from 0 to ∞, if not otherwise stated.

[3] For further references see G. H. Hardy, *Divergent series*, Oxford, 1949, notes on §6.1, p. 145.

and the same as

(6) $$\sum a_1^{(p)} = s\ (C_1 \text{ or } H_1) \qquad \text{for}^4\ p = k-1.$$

But from (2) and (4) it evidently follows that $\nu a_\nu^{(p)} \leqq M$ for all ν and $p=1, 2, \cdots$. Therefore, the theorem taken for granted for $k=1$, from (6) it follows that also

(7) $$\sum a_\nu^{(p)} = s$$

is true for $p=k-1$. Repeating the same argument we find that (7) is true also for $p=k-2, \cdots, 1, 0$ and Theorem II is proved.

To deduce Theorem I from it, we have to use two well known results of the general theory. First, the fact that $0 \leqq k < l$ and $\sum a_\nu = s\ (C_k)$ imply $\sum a_\nu = s\ (C_l)$. On account of this we may, in Theorem I too, suppose k integral and $\geqq 2$, replacing it by $l=[k]+1$ otherwise. And secondly the fact that C_k implies H_k—the easier half of the theorem of equivalence of the two methods (for integral k).

If we attack Theorem I directly the corresponding proof runs as follows: (1) with an integral $k \geqq 2$ is the same as

(8) $$\sum b_\nu^{(p+1)} = s \qquad \text{for } p = k-1$$

with

$$b_n^{(p+1)} = \frac{1}{n\binom{n+p+1}{n}} \sum_{\nu=0}^{n} \binom{n-\nu+p}{n-\nu} \nu a_\nu$$

($n \geqq 1$, $b_0^{(p+1)} = a_0$, $b_\nu^{(0)} = a_\nu$)—this being the C_{p+1}-series-to-series transform of $\sum a_\nu$. Instead of using now one part of the equivalence theorem of the C- and H-methods we have to use a similar result of the general theory, namely that C_k implies C_1C_{k-1}: If (8) is true for $p=k-1$ we have for the same p

(9) $$\sum b_\nu^{(p)} = s\ (C_1).$$

Now from (2) evidently it follows again that $\nu b_\nu^{(p)} \leqq M$ for all ν and $p=1, 2, \cdots$. Therefore, the theorem being true for the C_1-method, from (9) it follows that also

(10) $$\sum b_\nu^{(p)} = s$$

[4] The C_1-series-to-series transform of the series $\sum c_\nu$ is the series $c_0 + \sum_{\nu \geqq 1} (c_1 + 2c_2 + \cdots + \nu c_\nu)/\nu(\nu+1)$.

is true for $p=k-1$. Repeating, as before, the same argument the theorem follows.

Dr. Jurkat kindly pointed out to me that nearly the same argument yields the following more general

THEOREM III. *Let*

$$\alpha_n = \sum_\nu b_{n\nu} a_\nu \qquad (n = 0, 1, 2, \cdots) \tag{B}$$

be a series-to-series transform of the series $\sum a_\nu$ *into the series* $\sum \alpha_n$. *Let*

$$a_\nu = O(c_\nu) \tag{11}$$

be a Tauberian condition for the summation method B. Then the same condition (11) *is also a Tauberian condition for the method* $B \cdot B = B^2$ *and therefore for all iterated methods* $B^k = B \cdot B^{k-1}$ $(k=2, 3, \cdots)$ *if*

$$\sum_\nu | b_{n\nu} c_\nu | = O(c_n).$$

Furthermore, if B is regular and if (11) *is a best Tauberian condition for the method B, it is also a best condition for the method* B^k.

This follows at once from the inclusion $B^{k-1} \subset B^k$.

Finally, we remark that Theorem III and the last addition to it are evidently also true, the a_ν being real, for a one-sided Tauberian condition $a_\nu = O_R(c_\nu)$ if the $b_{n\nu}$ are $\geqq 0$ and if $\sum_\nu b_{n\nu} c_\nu = O_R(c_n)$.

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