KLOOSTERMAN'S METHOD IN TAUBERIAN THEOREMS FOR C_k SUMMABILITY[1]

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1. Most proofs of the Tauberian theorem of Hardy and Landau which states that

$$\text{(1)} \qquad \text{“}\sum a_n \textit{ is } C_k \textit{ summable”} + \text{“}na_n \leqq M\text{”} \to \text{“}\sum a_n \textit{ converges”}$$

(see [2, p. 121]) employ induction with respect to k, whereby the proof for arbitrary positive (whole) k is reduced to that for the case $k=1$. In the preceding paper Knopp [5] shows, however, that induction works much more smoothly in the case of iterative methods such as H_k summability. Knopp then points out that one might use this fact together with the equivalence of the C_k and H_k methods in the proof of (1): if $na_n \leqq M$, then $C_k \to H_k \to H_1 \to C_1$.

Kloosterman's method [3; 4] to prove (1) likewise has a more natural domain of application: it can be applied to integrals much more easily than to series as will be shown below. Moreover the analogue of (1) for integrals proved in this note contains (1) as a special case.

2. Let $a(t)$ be defined for $t \geqq 0$ and integrable over every finite interval $(0, x)$, $x>0$. The integral $\int a(t)dt$ is said to be C_k summable ($k \geqq 0$), to the value s, if

$$\text{(2)} \qquad \begin{aligned} A_k(x) &= \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_k} a(t)dt \\ &= \frac{x^k}{\Gamma(k+1)} \int_0^x a(t)(1 - t/x)^k dt \sim \frac{x^k}{\Gamma(k+1)} s \quad (x \to \infty) \end{aligned}$$

(compare [2, p. 110]). Here the first expression for $A_k(x)$ is of course to be used only if k is an integer.

With any series $\sum a_n$ one may associate the step function $a(t)$ defined by $a(t) = a_n$ ($n \leqq t < n+1$, $n = 0, 1, \cdots$). The argument in [2, pp. 113–114] may be used to show that the assertions $\sum a_n = s(C_k)$ and $\int a(t)dt = s(C_k)$ are equivalent. Thus (1) is contained in the following (known) result (compare [2, p. 135]).

THEOREM. *If $a(t)$ is real, if $k>0$ and*

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$$\int a(t)dt = s(C_k), \tag{3}$$

while

$$ta(t) \leqq M \qquad (t \geqq 0), \tag{4}$$

then

$$\int_0^\infty a(t)dt = s. \tag{5}$$

3. The proof will be based on a formula which expresses the mth derivative $f^{(m)}(x)$ of a function $f(x)$ in terms of the differences $\Delta^m f(x)$, $\Delta^{m+1}f(x), \cdots, \Delta^n f(x)$, where $\Delta f(x) = f(x+h) - f(x)$, etc. This formula occurs in the literature in a slightly weaker form, namely in A. A. Markoff's book [**6**, p. 21], and in Kloosterman's paper [**4**, Theorem 3]. Markoff's derivation can easily be modified so as to furnish the proof of the present lemma.

Lemma. *Let $f(x)$ be real and have derivatives of order* $1, 2, \cdots, n$ *on (a, b). Let the points x, $x+nh$ be contained in (a, b). Let $0 \leqq m \leqq n$. Then*

$$h^m f^{(m)}(x) = B_m^m \Delta^m f(x) + \cdots + B_m^n \Delta^n f(x) + B_m^{n+1} h^{n+1} \Phi, \tag{6}$$

where

$$B_m^p = \left\{ \frac{d^m}{dz^m} \binom{z}{p} \right\}_{z=0}, \tag{7}$$

while Φ can be written in the form

$$\Phi = \frac{f^{(n)}(x_2) - f^{(n)}(x_1)}{x_2 - x_1}, \tag{8}$$

with x_1, x_2 contained in the interval with end points x, $x+nh$.

4. Proof of the theorem. As in the preceding paper one may assume that k is an integer $\geqq 1$. Let $\epsilon > 0$ be given. By (3) one can determine x_0 such that

$$A_k(x) = (x^k/k!)s + \theta(x)\epsilon x^k \qquad (-1 < \theta(x) < 1 \text{ for } x > x_0).$$

Now apply the lemma to $f(x) = A_k(x)$, taking $n = m = k$, $x > x_0$, $0 < h < x/k$. Note that, by (7),

$$B_k^k = 1, \; B_k^{k+1} = -k/2,$$

while by the definition of Δ^k,

$$\begin{aligned}\Delta^k A_k(x) &= h^k s + \epsilon \sum (-1)^{k-j} \binom{k}{j} \theta(x+jh)(x+jh)^k \\ &> h^k s - \epsilon \sum \binom{k}{j} (2x)^k = h^k s - \epsilon \cdot 4^k x^k.\end{aligned}$$

Furthermore by (8) and (4)

$$\Phi = \left\{ \int_{x_1}^{x_2} a(t)dt \right\} \Big/ (x_2 - x_1) \leqq M/x.$$

Collecting results, one obtains from (6) the estimate

$$\begin{aligned}\int_0^x a(t)dt &= h^{-k}\Delta^k A_k(x) = 2^{-1}kh\Phi \\ &> s - \epsilon \cdot 4^k (x/h)^k - 2^{-1}kMh/x \qquad (x > x_0).\end{aligned} \tag{9}$$

Choosing

$$h = \epsilon^{1/(k+1)} x$$

—for this choice of h, kh will be less than x provided ϵ is small enough —(9) takes the form

$$\int_0^x a(t)dt > s - C\epsilon^{1/(k+1)} \qquad (x > x_0), \tag{10}$$

where C is independent of ϵ. An estimate in the opposite direction may be obtained by taking, from the beginning, $0 > h > -2^{-1}x/k$. This completes the proof of (5).

5. In the above proof formula (6) was used only in the special case $m = n = k$ (compare [3]). Various other (known) results may be derived from (6). For example, one finds by taking $f(x) = A_k(x)$, $0 < m \leqq n = k$, that the hypotheses $a(x) = A_{-1}(x) < Mx^\alpha$ and $A_k(x) = o(x^\beta)$ (with $\beta - \alpha \leqq k+1$) imply the "convexity" estimate $A_{k-m}(x) = o(x^\gamma)$ where $\gamma = \{m\alpha + (k+1-m)\beta\}/(k+1)$ (compare [1], where more general estimates are proved). Another example is the Tauberian theorem on C_k summable gap series [7].

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