A NOTE ON THE HAMBURGER AND STIELTJES MOMENT PROBLEMS

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1. **Introduction.** In 1939 Boas[1] proved that if $\mu_0 \geqq 1$, $\mu_n \geqq (n\mu_{n-1})^n$ $(n=1, 2, \cdots)$, then the Stieltjes moment problem

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, \cdots) \tag{1.1}$$

has a nondecreasing solution $\alpha(t)$ with infinitely many points of increase; moreover, if $\lambda_0 \geqq 1$, $\lambda_2 \geqq (2\lambda_1+2)^2$, $\lambda_n \geqq (n\lambda_{n-1})^n$ $(n=1, 3, 4, 5, \cdots)$, and $\mu_n = \lambda_{2n}$ $(n=0, 1, 2, \cdots)$, then the moment problem (1.1) is indeterminate.

It is the purpose of the present note to show that under much weaker conditions on the rate of growth of the sequence $\{\mu_n\}$ one has for (1.1) the existence of a nondecreasing solution with infinitely many points of increase and that the problem is indeterminate. Preliminary to the proof of this result a corresponding result for the Hamburger moment problem is established.

The principal result of the present note is contained in the following two theorems.

THEOREM A. *If $\{\mu_n\}$ is a sequence of real numbers for which there exists a monotone decreasing sequence of positive numbers $\{a_n\}$ such that*

$$\mu_0 \geqq a_0, \ \mu_{2n} \geqq a_n + (a_{n-1} - a_n)^{-1} \sum_{j=0}^{n-1} \mu_{n+j}^2 \qquad (n = 1, 2, \cdots), \tag{1.2}$$

then the Hamburger moment problem

$$\mu_n = \int_{-\infty}^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, \cdots) \tag{1.3}$$

has a nondecreasing solution $\alpha(t)$ with infinitely many points of increase. Moreover, if $a = \lim_{n\to\infty} a_n$ is positive, then the moment problem (1.3) *is indeterminate.*

THEOREM B. *If in addition to condition* (1.2) *there is a monotone*

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[1] Boas' proof of this result is given in D. V. Widder, *The Laplace transform*, Princeton University Press, 1946, pp. 140–142.

decreasing sequence of positive numbers $\{b_n\}$ *such that*

$$(1.4)\quad \mu_1 \geqq b_0,\ \mu_{2n+1} \geqq b_n + (b_{n-1} - b_n)^{-1} \sum_{j=0}^{n-1} \mu_{1+n+j}^2 \quad (n = 1, 2, \cdots),$$

then the Stieltjes moment problem (1.1) *has a nondecreasing solution* $\alpha(t)$ *with infinitely many points of increase. Moreover, if* $a = \lim_{n\to\infty} a_n$ *and* $b = \lim_{n\to\infty} b_n$ *are positive, then the moment problem* (1.1) *is indeterminate.*

2. **Proofs of Theorems A and B.** The Hamburger moment problem (1.3) has a nondecreasing solution $\alpha(t)$ with infinitely many points of increase[2] if and only if the sequence $\{\mu_n\}$ is positive definite, i.e.

$$(2.1)\qquad Q_n[x] \equiv \sum_{i,j=0}^{n} \mu_{i+j} x_i x_j \qquad (n = 0, 1, 2, \cdots)$$

are positive definite quadratic forms. Consequently, the first part of the theorem is established if it is shown that condition (1.2) implies

$$(2.2)\qquad Q_n[x] \geqq a_n \sum_{j=0}^{n} x_j^2 \qquad (n = 0, 1, 2, \cdots),$$

which will be accomplished by mathematical induction. Clearly the condition $\mu_0 \geqq a_0$ implies (2.2) for $n=0$. Now

$$(2.3)\quad Q_{m+1}[x] = Q_m[x] + 2\left(\sum_{j=0}^{m} \mu_{m+1+j} x_j \right) x_{m+1} + \mu_{2m+2} x_{m+1}^2,$$

and if (2.2) holds for $n=m$ then the relation (1.2) for $n=m+1$ implies that the right-hand member of (2.3) is not less than

$$\begin{aligned} &a_m \sum_{j=0}^{m} x_j^2 + 2\left(\sum_{j=0}^{m} \mu_{m+1+j} x_j \right) x_{m+1} \\ &\qquad + \left[a_{m+1} + (a_m - a_{m+1})^{-1} \sum_{j=0}^{m} \mu_{m+1+j}^2 \right] x_{m+1}^2 \\ &= a_{m+1} \left(\sum_{j=0}^{m+1} x_j^2 \right) + (a_m - a_{m+1})^{-1} \sum_{j=0}^{m} \left[(a_m - a_{m+1}) x_j + \mu_{m+1+j} x_{m+1} \right]^2 \\ &\geqq a_{m+1} \sum_{j=0}^{m+1} x_j^2, \end{aligned}$$

[2] H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann. I. vol. 81 (1920) pp. 235–319; II. vol. 82 (1921) pp. 120–164; III. vol. 82 (1921) pp. 168–187; in particular, Chapter 2 of I. See also Widder, loc. cit. pp. 134–136.

so that (2.2) holds for $n=m+1$ also.

To establish the last statement of Theorem A, it is to be noted that if the sequence $\{\mu_n\}$ satisfies (1.2) with the sequence $\{a_n\}$, then the sequence $\{\mu_n^*\}=\{\mu_{n+2}\}$ satisfies (1.2) with the sequence $\{a_n^*\}=\{a_{n+1}\}$. Clearly (1.2) implies that $\mu_0^*=\mu_2\geqq a_1=a_0^*$, and for $n\geqq 1$ that

$$\begin{aligned}\mu_{2n}^* = \mu_{2(n+1)} &\geqq a_{n+1} + (a_n - a_{n+1})^{-1}\sum_{i=0}^{n} \mu_{n+1+i}^2 \\ &\geqq a_{n+1} + (a_n - a_{n+1})^{-1}\sum_{i=1}^{n} \mu_{n+1+i}^2 \\ &= a_n^* + (a_{n-1}^* - a_n^*)^{-1}\sum_{j=0}^{n-1} \mu_{n+j}^{*2}.\end{aligned}$$

From the relation (2.2) applied to $\{\mu_n^*\}$ it then follows that

$$(2.4)\qquad Q_n^{(2)}[x] \equiv \sum_{i,j=0}^{n} \mu_{2+i+j}x_i x_j \geqq a_{n+1}\sum_{j=0}^{n} x_j^2 \qquad (n = 0, 1, 2, \cdots).$$

If M_n and $M_n^{(2)}$ denote the respective minima of $Q_n[x]$ and $Q_n^{(2)}[x]$ on the plane $x_0=1$, then by a result of Hamburger[3] the moment problem (1.3) is determined if and only if at least one of the sequences $\{M_n\}$, $\{M_n^{(2)}\}$ converges to zero. Consequently, from (2.2) and (2.4) it follows that the Hamburger moment problem is indeterminate if $a=\lim_{n\to\infty} a_n>0$.

In view of the proof of Theorem A the hypotheses of Theorem B imply (2.2) and

$$Q_n^{(1)}[x] \equiv \sum_{i,j=0}^{n} \mu_{1+i+j}x_i x_j \geqq b_n\sum_{j=0}^{n} x_j^2 \qquad (n = 0, 1, \cdots).$$

Consequently, the results of Theorem B are immediate consequences of the following facts: (i) the Stieltjes moment problem (1.1) has a nondecreasing solution $\alpha(t)$ with infinitely many points of increase if and only if the sequences $\{\mu_n\}$ and $\{\mu_{n+1}\}$ are positive definite;[4] (ii) if M_n and $M_n^{(1)}$ denote the respective minima of $Q_n[x]$ and $Q_n^{(1)}[x]$ on the plane $x_0=1$, then (1.1) is determined if and only if at least one of the sequences $\{M_n\}$, $\{M_n^{(1)}\}$ converges to zero.[5]

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[3] See Hamburger, loc. cit., in particular, Theorem XXX of III, or J. A. Shohat and J. D. Tamarkin, *The problem of moments*, Mathematical Surveys, No. 1, American Mathematical Society, 1943, p. 70.

[4] See T. J. Stieltjes, *Recherches sur les fractions continues*, Annales de la faculté des sciences de Toulouse vol. 8 (1894) pp. 1–122, or Widder, loc cit. pp. 136–138.

[5] See Hamburger, loc. cit., in particular, Theorem XXXI of III, or Shohat and Tamarkin, loc. cit., p. 75.

3. **Additional comments.** It is to be remarked that in (1.2) the odd moments enter only in the right-hand members of the inequalities. In particular, if $\{\mu_n\}$ satisfies (1.2) with a sequence $\{a_n\}$, and $\mu'_{2n}=\mu_{2n}$, $|\mu'_{2n+1}| \leqq |\mu_{2n+1}|$, $(n=0, 1, \cdots)$, then $\{\mu_n'\}$ satisfies (1.2) with the same sequence $\{a_n\}$.

If $\{\mu_n\}$ is a sequence of real numbers satisfying the conditions

$$(3.1) \qquad \mu_0 \geqq c_0,\ \mu_1 \geqq c_1,\ \mu_n \geqq c_n + (c_{n-1} - c_n)^{-1} \sum_{i=[(n+1)/2]}^{n-1} \mu_i^2 \qquad (n = 2, 3, \cdots),$$

where $\{c_n\}$ is a monotone decreasing sequence of positive numbers, then it may be verified readily that (1.2) holds for $\{a_n\} = \{c_{2n}\}$ and (1.4) holds for $\{b_n\} = \{c_{2n+1}\}$. In particular, if $\{\mu_n\}$ is a nondecreasing sequence satisfying with a monotone decreasing sequence of positive numbers $\{c_n\}$ the conditions

$$(3.2) \qquad \mu_0 \geqq c_0,\ \mu_1 \geqq c_1,\ \mu_n \geqq c_n + (c_{n-1} - c_n)^{-1}(n/2)\mu_{n-1}^2 \qquad (n = 2, 3, \cdots),$$

then $\{\mu_n\}$ satisfies with $\{c_n\}$ the conditions (3.1).

As a special case, if $\{\mu_n\}$ satisfies the conditions $\mu_0 \geqq 1$, $\mu_n \geqq (n\mu_{n-1})^n$ $(n=1, 2, \cdots)$ of Boas, then $\{\mu_n\}$ is monotone nondecreasing and satisfies (3.2) with

$$c_0 = 1,\ 3/4 < c_1 < 1,\ c_n = c_1 + 2^{-n} - 3/4 \qquad (n = 2, 3, \cdots),$$

since $\mu_2 \geqq 4\mu_1^2 > c_2 + 2\mu_1^2 = c_2 + (c_1 - c_2)^{-1}\mu_1^2$, while for $n \geqq 3$ we have $\mu_n \geqq n^n \mu_{n-1}^n > n^n \mu_{n-1}^2 > 1 + (n^n - 1)\mu_{n-1}^2 > 1 + n(n-1)^{n-1}\mu_{n-1}^2 > c_n + n2^{n-1}\mu_{n-1}^2 = c_n + (c_{n-1} - c_n)^{-1}(n/2)\mu_{n-1}^2$. Since $c_n \to c_1 - 3/4 > 0$ as $n \to \infty$, from Theorem B it follows that if $\mu_0 \geqq 1$, $\mu_n \geqq (n\mu_{n-1})^n$ $(n=1, 2, \cdots)$, then the Stieltjes moment problem (1.1) has a nondecreasing solution with infinitely many points of increase; moreover, from the last part of this theorem it follows that (1.1) is indeterminate for this moment sequence, as well as for an allied sequence with a higher rate of increase as in the example of Boas.

It is to be remarked that if $0 < r < 1$ and

$$(3.3) \qquad \mu_0 = 1,\ \mu_n \geqq 1 - r + r^{n+1} + n\mu_{n-1}^2/(2r^n(1 - r)) \qquad (n = 1, 2, \cdots),$$

it follows readily that $\{\mu_n\}$ is a monotone increasing sequence satisfying (3.1) with $c_n = 1 - r + r^{n+1}$ $(n=0, 1, \cdots)$, and $c_n \to 1 - r$ as $n \to \infty$, so that the Stieltjes moment problem (1.1) admits a nondecreasing solution with infinitely many points of increase and the problem is indeterminate. Since for $0<r<1$ we have $1/(2r^n(1-r)) >1>1-r+r^{n+1}$ $(n=1, 2, \cdots)$, it follows that (3.3) holds if

$$\mu_0 = 1,\ \mu_n \geqq (1 + n\mu_{n-1}^2)/(2r^n(1 - r)) \qquad (n = 1, 2, \cdots),$$

which in turn is valid if

$$\mu_0 = 1,\ \mu_n \geqq \mu_{n-1}^2\{(1 + n)/(2r^n(1 - r))\} \qquad (n = 1, 2, \cdots).$$

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