REPRESENTATIONS OF RESTRICTED LIE ALGEBRAS OF CHARACTERISTIC p

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It was proved by Jacobson [2] that every finite-dimensional Lie algebra of characteristic p has a finite-dimensional representation space which is not semisimple. In this note we prove a similar result for Jacobson's restricted Lie algebras [1].

The structure of a restricted Lie algebra L over a field F of characteristic p comprises, in addition to the ordinary Lie algebra structure, a map $x \to x^{[p]}$ of L into itself, with formal properties corresponding to those of the map $x \to x^p$ in an associative algebra of characteristic p. Accordingly, one studies the "restricted" representations of L, in which the operator corresponding to $x^{[p]}$ is the pth power of the operator corresponding to x. Our result is the following.

THEOREM. *Let L be a finite-dimensional restricted Lie algebra over a field F of characteristic p. A necessary and sufficient condition for all finite-dimensional restricted representation spaces of L to be semisimple is that L be abelian and the elements $x^{[p]}$ span L over F.*

We imbed L in its u-algebra, as defined in [1]; U_L, say. U_L is an associative algebra over the basic field F which contains L as a subspace and is generated by the elements of L. If x and y are elements of L, and $[x, y]$ denotes their commutator in L, then one has, in U_L, $xy-yx=[x, y]$, and $x^p=x^{[p]}$. In fact, these are precisely the defining relations by means of which U_L is obtained as a homomorphic image of the tensor algebra over L. If $x_1, \cdots, x_n$ is a basis for L over F, every element of U_L can be written in one and only one way as an F-linear combination of the ordered monomials $x_1^{e_1} \cdots x_n^{e_n}$, with $0 \leqq e_i < p$, and not all $e_i=0$. A restricted representation space of L is simply a representation space of U_L.

Now suppose first that L is abelian and the elements $x^{[p]}$ span L over F. Then, if $y_i=x_i^{[p]}$, $y_1, \cdots, y_n$ is a basis for L over F, and we have $x_i^p=y_i$, in U_L. Furthermore, U_L is commutative, in this case. Let u be any nonzero element of U_L. Write u as a linear combination of ordered monomials in the x_i. Then u^p is the linear combination of ordered monomials in the y_i which is obtained from u by replacing each x_i by y_i, and each coefficient in F by its pth power. Hence $u^p \neq 0$. Therefore U_L has no nilpotent elements other than 0, and since U_L is finite-dimensional and commutative this implies that

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U_L is semisimple, and so that every finite-dimensional representation space of U_L is semisimple. This proves the sufficiency of the condition of our theorem.

Next we shall construct a representation space of U_L which cannot be semisimple unless the condition of the theorem is satisfied. Let M denote a vector space of dimension n over F, and let $t_1, \cdots, t_n$ be a basis for M over F. Let f be the p-semilinear map of L into M defined by $f(a_1x_1 + \cdots + a_nx_n) = a_1^p t_1 + \cdots + a_n^p t_n$, where $a_i \in F$. We construct a Lie algebra E whose underlying space is the direct sum of M and the space of L. The elements of E will be written as pairs (x, m), with $x \in L$ and $m \in M$. The commutation in E is defined by $[(x, m), (x', m')] = ([x, x'], 0)$. We make E into a restricted Lie algebra by setting $(x, m)^{[p]} = (x^{[p]}, f(x))$. Indeed, since $(0, M)$ lies in the center of E and since f is p-semilinear, this map evidently satisfies the requirements for a p-map in a restricted Lie algebra.

Now we consider the u-algebra U_E of E. The homomorphism $(x, m) \to x$ of E onto L is evidently compatible with the p-maps and can therefore be extended uniquely to a homomorphism ϕ of U_E onto U_L. The kernel of ϕ is evidently the ideal M' of U_E which is generated by the elements of $(0, M) = M$. If we write the elements of U_E as linear combinations of ordered monomials in the elements of a basis of E which contains a basis of M we see that, as a vector space, M' is the direct sum of its subspaces U_EM and M. Let γ denote the corresponding projection of M' onto M, i.e., γ maps U_EM into (0) and leaves the elements of M fixed.

The map $x \to (x, 0)$ of L into E can evidently be extended to a linear map ψ of U_L into U_E such that $\phi\psi$ is the identity map on U_L. Then, if u and v are arbitrary elements of U_L, $\psi(u)\psi(v) - \psi(uv)$ lies in the kernel M' of ϕ, and we can define a bilinear map g of (U_L, U_L) into M by setting $g(u, v) = \gamma(\psi(u)\psi(v) - \psi(uv))$. It is easily verified that $g(uv, w) = g(u, vw)$, for all u, v, w in U_L.

Now we construct a representation of U_L as follows. The space S of the representation is the direct sum of M and the space of U_L. We write $S = U_L + M$, and correspondingly write the elements of S in the form $v + m$, where $v \in U_L$ and $m \in M$. If $u \in U_L$ we define the linear transformation $s \to u \cdot s$ of S that corresponds to u by setting $u \cdot (v + m) = uv + g(u, v)$. Since g is bilinear and associative, this defines the structure of a representation space for U_L on S.

Now suppose that S is semisimple. Then the U_L-invariant subspace M of S must have a U_L-invariant complement; $S = Q + M$, where Q is a U_L-invariant subspace of S and $Q \cap M = (0)$. For every $v \in U_L$ we have therefore $v = q(v) + h(v)$, where $q(v) \in Q$ and $h(v) \in M$. Evidently,

h is a linear map of U_L into M. Furthermore, if we operate on v with $u \in U_L$, we find that $h(uv) = -g(u, v)$.

Now we observe that for any x and y in L we have $g(x, y) = \gamma((x, 0)(y, 0) - \psi(xy)) = \gamma((x, 0)(y, 0) - \psi([x, y]) - \psi(yx)) = \gamma((y, 0) \cdot (x, 0) - \psi(yx)) = g(y, x)$. Hence $h([x, y]) = h(xy) - h(yx) = 0$, so that h vanishes on the derived algebra $[L, L]$ of L. On the other hand,

$$\begin{aligned} h(x^{[p]}) &= h(x^p) = -g(x^{p-1}, x) = \gamma(\psi(x^p) - \psi(x^{p-1})\psi(x)) \\ &= \gamma((x^{[p]}, 0) - \psi(x)^p) \\ &= \gamma((x^{[p]}, 0) - (x, 0)^{[p]}) = -f(x). \end{aligned}$$

Now let a_i be arbitrary elements of F. Then we have, by the definition of f, and the last result, $h(\sum_{i=1}^n a_i x_i^{[p]}) = -\sum_{i=1}^n a_i t_i$. Since h vanishes on $[L, L]$ and the t_i are linearly independent, this shows that the elements $x_i^{[p]}$ are linearly independent mod $[L, L]$. This can be the case only if $[L, L] = (0)$ and the $x_i^{[p]}$ are linearly independent. This completes the proof of our theorem.

References

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