A NOTE ON LINEAR VECTOR SPACES OF MAPPINGS WITH POSITIVE JACOBIANS

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Let $\mathfrak{F}$ be a family of continuously differentiable mappings f of an open set D in the xy-plane into the uv-plane satisfying the following conditions:

(a) $f, g \in \mathfrak{F} \rightarrow \lambda f + \mu g \in \mathfrak{F}$ for all real λ, μ,

(b) $f \in \mathfrak{F} \rightarrow$ the Jacobian of f is non-negative and zero only if its rank is zero,

(c) $\mathfrak{F}$ contains two mappings $f_1 = (u_1, v_1)$, $f_2 = (u_2, v_2)$ such that the Jacobian

$$\frac{\partial(v_1, v_2)}{\partial(x, y)} \neq 0$$

for $(x, y) \in D$.

The mappings described by functions $f = u + iv$ regular analytic in $z = x + iy$, $(x, y) \in D$, form such a family. J. E. McLaughlin and C. J. Titus [1] proved the converse: If the two mappings of condition (c) are described by functions regular analytic in $z = x + iy$, then $\mathfrak{F}$ is a linear vector space of functions regular analytic in $z = x + iy$, $(x, y) \in D$. It is the purpose of this note to give a characterization of the general class $\mathfrak{F}$.

Consider continuously differentiable solutions $u(x, y)$, $v(x, y)$ of the system of partial differential equations

$$(1) \qquad L_1 \equiv u_x - a_{11}v_x - a_{12}v_y = 0, \qquad L_2 \equiv u_y + a_{21}v_x + a_{22}v_y = 0$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are real functions of x, y continuous in D. If the system is elliptic [2], $(a_{11} + a_{22})^2 < 4a_{12}a_{21}$ and $a_{12} > 0$, $a_{21} > 0$ in D, then the Jacobian

$$\frac{\partial(u, v)}{\partial(x, y)} = a_{21}v_x^2 + (a_{11} + a_{22})v_xv_y + a_{12}v_y^2$$

is positive unless it is of rank 0. Therefore, the mappings f described by functions u, v continuously differentiable in D which are solutions of the elliptic system (1) form a family $\mathfrak{F}$. We prove the converse.

THEOREM. *The mappings f of a family $\mathfrak{F}$ are described by functions u, v that are solutions of a unique elliptic system of form* (1).

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To prove this consider the mappings f_1, f_2 satisfying condition (c). Then we can solve the system

$$u_{ix} - a_{11}v_{ix} - a_{12}v_{iy} = 0, \quad u_{iy} + a_{21}v_{ix} + a_{22}v_{iy} = 0 \qquad (i = 1, 2)$$

for a_{11}, a_{12}, a_{21}, a_{22}, and the solution is uniquely determined. The system

$$L_1 \equiv u_x - a_{11}v_x - a_{12}v_y = 0, \quad L_2 \equiv u_y + a_{21}v_x + a_{22}v_y = 0$$

is then satisfied by the linear combinations (u, v) of (u_1, v_1), (u_2, v_2) with real constant coefficients, and since the corresponding mappings f are in $\mathfrak{F}$ we have by (b)

$$\frac{\partial(u, v)}{\partial(x, y)} = a_{21}v_x^2 + (a_{11} + a_{22})v_xv_y + a_{12}v_y^2 > 0$$

except where $v_x = v_y = 0$. Since every real vector (X, Y) is a linear combination with real coefficients of the linearly independent vectors (v_{1x}, v_{1y}), (v_{2x}, v_{2y}), the form $a_{21}X^2 + (a_{11} + a_{22})XY + a_{12}Y^2$ is positive definite, $(a_{11} + a_{22})^2 < 4a_{12}a_{21}$, and $a_{12} > 0$, $a_{21} > 0$.

Now let $f = (u, v)$ be any mapping in class $\mathfrak{F}$. By Lemma 2 of [1] we know that

$$J(f) = \lambda_1 J(f_1) + \lambda_2 J(f_2)$$

where $J(f)$, $J(f_1)$, $J(f_2)$ are the Jacobian matrices of f, f_1, f_2 and λ_1, λ_2 are scalars that may depend on x, y. It follows that u, v satisfy the system

$$L_1 = 0, \qquad L_2 = 0,$$

and the theorem is proved.

REMARK 1. If the two mappings of condition (c) are described by functions regular analytic in $z = x + iy$, then $L_1 = 0$, $L_2 = 0$ are the Cauchy-Riemann equations and we obtain the characterization of the analytic functions given in [1].

REMARK 2. If the two mappings of condition (c) are described by sigma-monogenic functions [3] $f = u + iv$, where

$$\sigma_1(x)u_x = \tau_1(y)v_y, \qquad \sigma_2(x)u_y = -\tau_2(y)v_x$$

with $\sigma_i > 0$, $\tau_i > 0$, then $\mathfrak{F}$ contains only sigma-monogenic functions.

REMARK 3. If the two mappings of condition (c) are described by functions u_i, v_i analytic in the real variables x, y, then the coefficients a_{ij} of the system $L_1 = 0$, $L_2 = 0$ are analytic, and by a well known property of such systems, $\mathfrak{F}$ contains only mappings described by functions analytic in x, y.

References

1. J. E. McLaughlin, and C. J. Titus, *A characterization of analytic functions*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 348–351.

2. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. II, Berlin, 1937; see p. 143.

3. Lipman Bers, *The expansion theorem for sigma-monogenic functions*, Amer. J. Math. vol. 72 (1950) pp. 705–712.

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