References

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ON A PROBLEM OF ADDITIVE NUMBER THEORY

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Let A, B, $\cdots$ denote sets of natural numbers. *The counting function* $A(n)$ *of* A is the number of elements $a \in A$ which satisfy the inequality $a \leqq n$. We shall call two sets A, B *complementary* to each other if $A+B$ contains all sufficiently large natural numbers.

In a talk with the author P. Erdös conjectured that each infinite set A has a complementary set B of asymptotic density zero. Here we wish to establish a theorem which gives an upper estimate for $B(n)$ in terms of $A(n)$. As a particular case, the truth of Erdös' conjecture will follow. The estimate (1) below should be compared with the (trivial) lower estimate $B(n) \geqq (1-\epsilon)n/A(n)$, which holds for all large n.

Theorem 1. *For each infinite set A there is a complementary set B such that*

$$B(n) \leqq C \sum_{k=1}^{n} \frac{\log A(k)}{A(k)}; \tag{1}$$

C is an absolute constant and the terms of the sum with $A(k)=0$ *are to be replaced by one.*

Proof. Let A be given and let $m<n$ denote two natural numbers. We shall choose certain integers b in the interval $m \leqq b < 2n$ in such a way that the sums $a+b$, $a \in A$, fill the whole interval $n < a+b \leqq 2n$. Our concern will be to obtain the upper estimate (4) for the number K of the b's.

First we take a b_1 in $[m, 2n)$ in such a way that the portion of $A+b_1$ contained in $(n, 2n]$ has the maximal possible number S of elements and choose this b_1 as one of our b's. Then we take another

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b_2 in $[m, 2n)$ such that $A+b_2$ contains the maximal possible number of integers in $(n, 2n]$ which do not belong to $A+b_1$, and so on. To each $s \leqq S \leqq n$ there will correspond a certain number $K_s \geqq 0$ of translations of A of the form $A+b$, $m \leqq b < 2n$, such that the portion of $A+b$ contained in $(n, 2n]$ has exactly s new points. We shall say that b corresponds to the value s.

Let R denote the set of integers in $(n, 2n]$ which are not covered after all elements b corresponding to the values $S, S-1, \cdots, s+1$ are already chosen. Let $r_1 < r_2 < \cdots < r_k$, $k = k_s$, denote the elements of R. Clearly, $k_0 = 0$ and

$$sK_s = k_s - k_{s-1}, \qquad s = 1, 2, \cdots. \tag{2}$$

On the other hand, an upper bound for k_s is easily obtained. Consider the translations $A+m, \cdots, A+2n-1$ of A and count the number of times the points of R are covered by these translations. Each translation contains at most s points of R, hence the number of coverings does not exceed $2ns$. On the other hand, a point $r_i \in R$ is covered exactly $A(r_i - m)$ times. Hence

$$A(r_1 - m) + A(r_2 - m) + \cdots + A(r_k - m) \leqq 2ns,$$

and $kA(n-m+1) \leqq 2ns$, so that

$$k_s \leqq 2s \frac{n}{A(n-m+1)}. \tag{3}$$

From this we derive an upper bound for K. Let s_0 be a natural number; then the number of b's corresponding to values $s < s_0$ is $\sum_{s=1}^{s_0-1} K_s$. On the other hand, the number of b's corresponding to values $s \geqq s_0$ cannot exceed n/s_0, since each choice of a b introduces at least s_0 new points in $(n, 2n]$. Hence, by (2) and (3),

$$\begin{aligned} K &\leqq \sum_{s=1}^{s_0-1} K_s + \frac{n}{s_0} \\ &\leqq \sum_{s=1}^{s_0} \frac{1}{s}(k_s - k_{s-1}) + \frac{n}{s_0} \\ &= \sum_{s=1}^{s_0-1} \frac{1}{s(s+1)} k_s + \frac{1}{s_0} k_{s_0} + \frac{n}{s_0} \\ &\leqq \frac{2n}{A(n-m+1)} \sum_{s=1}^{s_0} \frac{1}{s} + \frac{n}{s_0} \\ &\leqq \frac{C_1 n \log s_0}{A(n-m+1)} + \frac{n}{s_0}. \end{aligned}$$

Now we take $s_0=[A(n-m+1)/\log A(n-m+1)]$, then

$$K \leqq C_2 \frac{n \log A(n-m+1)}{A(n-m+1)} \tag{4}$$

(the right-hand side is to be replaced by $C_2 n$ if s_0 happens to be zero).

We now proceed to construct the set B. For each $l=1, 2, \cdots$ we take $n=n(l)=2^l$, $m=m(l)=2^{l-1}+1$, $K=K(l)$, denote the constructed set of b's in $(2^{l-1}, 2^{l+1})$ by B_l, and put $B=\cup B_l$. By (4),

$$K(l) = C \frac{2^{l-1} \log A(2^{l-1})}{A(2^{l-1})} \cdot \tag{5}$$

For an arbitrary N we have, by (5), since $B_l(N)=0$ for $N \leqq 2^{l-1}$,

$$B(N) \leqq \sum_l B_l(N) \leqq \sum_{2^{l-1} \leqq N} K(l) \leqq C \sum_{2^{l-1} \leqq N} \sum_{2^{l-2} < k \leqq 2^{l-1}} \frac{\log A(k)}{A(k)}$$
$$= C \sum_{k=1}^{N} \frac{\log A(k)}{A(k)},$$

which proves the theorem.

Since for an infinite set A the quotient $\log A(k)/A(k)$ tends to zero for $k\rightarrow\infty$, it follows from (1) that $B(n)=\sigma(n)$. This proves the conjecture of Erdös. Several other simple estimates follow immediately from (1). Of these, we give only the following. Let the set A have lower exponential density $\alpha>0$, that is, let

$$\liminf_{n\rightarrow\infty} \frac{\log A(n)}{\log n} = \alpha > 0.$$

Then there is a complementary set B with upper exponential density not exceeding $1-\alpha$, that is, such that

$$\limsup_{n\rightarrow\infty} \frac{\log B(n)}{\log n} \leqq 1 - \alpha.$$

This leads to the following remark. The well known theorem of Mann states that the addition of an arbitrary set B to a given set A leads in general only to the addition of lower densities. If, however, the set B is chosen in a proper way, depending on the set A, we can achieve the addition of exponential densities.

It is very likely that (1) is the best possible estimate if the rate of increase of $A(n)$, but not the structural properties of A, are taken into account. If $A(n)\geqq\alpha n$, (1) gives $B(n)\leqq C \log^2 n$, and Erdös has shown by a probabilistic argument (in a paper to appear in these Proceedings) that this cannot be improved. Also Theorem 2 below may be shown to be the best possible.

Finally, we give another application of formula (4). Taking $m=1$, we see that if $a_1, \cdots, a_l$ is an arbitrary set of natural numbers not exceeding $2n$, there is another set $b_1, \cdots, b_k$, $0<b_j<2n$, so that $k \leqq C_2 n \log l/l$ and that all integers $n<x\leqq 2n$ are of the form $x=a_i+b_j$. Reducing a's and b's modulo n we obtain:[1]

THEOREM 2. *If $a_1, \cdots, a_l$ is a set of incongruent residues modulo n, there is another set of residues $b_1, \cdots, b_k$ with*

$$k \leqq C \frac{n \log l}{l} \tag{6}$$

such that each residue modulo n is of the form a_i+b_j.

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[1] I am obliged to Erdös for the suggestion to apply my argument to this problem. In the case when n is a prime, he obtained a similar estimate by another method.