# ON A PROPOSITION OF SIERPIŃSKI'S WHICH IS EQUIVALENT TO THE CONTINUUM HYPOTHESIS

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W. Sierpiński obtained [**2**, p. 168; French translation **3**, p. 72] the following paradoxical theorem as a consequence of another result of his, and then published [**4**; reproduced in **5**, pp. 12–14] a direct proof.

THEOREM (SIERPIŃSKI). *If* $2^{\aleph_0}=\aleph_1$, *there exists an infinite sequence of single-valued functions of a real variable* $f_0(x), f_1(x), f_2(x), \cdots$ *such that, for every nondenumerable set* $N$ *of real numbers, all the functions of the sequence, except perhaps a finite number of them, transform* $N$ *onto the set of all real numbers.*

An examination of the direct proof shows, however, that all that can be said about the sequence of functions $f_0(x), f_1(x), f_2(x), \cdots$ defined there is that infinitely many of them, but not necessarily all except perhaps a finite number, transform $N$ onto the set of all real numbers. For if the sequence (see [**5**, p. 13]) $\xi_0^\alpha, \xi_1^\alpha, \xi_2^\alpha, \cdots$ is defined, e.g., in such a manner that, for every $\alpha<\Omega$, $\xi_{2k}^\alpha=2k$ $(k=0, 1, 2, \cdots)$, which the proof evidently allows, then each of the (infinitely many) functions $f_{2k}(x)$ transforms the set of all real numbers onto a single real number. Thus, further restrictions must be placed on the sequences $\xi_0^\alpha, \xi_1^\alpha, \xi_2^\alpha, \cdots$ $(\alpha<\Omega)$ in order that the sequence of functions $f_0(x), f_1(x), f_2(x), \cdots$ they are used to define possess the property required in the theorem. It is the purpose of this note to show how such restrictions can be made. Actually we shall prove the following somewhat stronger result:

THEOREM 1. *If* $2^{\aleph_0}=\aleph_1$, *there exists an infinite sequence of single-valued functions of a real variable* $f_0(x), f_1(x), f_2(x), \cdots$ *such that, for every nondenumerable set* $N$ *of real numbers, every function of the sequence, except perhaps a finite number of them, transforms* $N$ *onto the set of all real numbers in such a manner that every real number is the image of nondenumerably many elements of* $N$.

Let us assume, then, that $2^{\aleph_0}=\aleph_1$, and, as in [**5**, p. 13], well-order the set of all real numbers to form a transfinite sequence

$$x_\omega, x_{\omega+1}, x_{\omega+2}, \cdots, x_\alpha, \cdots \qquad (\omega \leqq \alpha < \Omega).$$

There exists a one-to-one correspondence between the set of all ordinal numbers less than $\Omega$ and the set of all infinite sequences of real

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numbers; for every $\alpha<\Omega$, we shall denote by $(t_0^\alpha, t_1^\alpha, t_2^\alpha, \cdots)$ the sequence of real numbers associated with $\alpha$ under this correspondence. The cardinal number of the set of all increasing infinite sequences of non-negative integers is $2^{\aleph_0}$, and we may therefore well-order this set to form a transfinite sequence

$$s_0, s_1, s_2, \cdots, s_\beta, \cdots \qquad (\beta < \Omega). \tag{1}$$

Now, for every $\alpha$ satisfying $\omega \leqq \alpha < \Omega$, we define the infinite sequence of ordinal numbers $\xi_0^\alpha, \xi_1^\alpha, \xi_2^\alpha, \cdots$ as follows. The power of the set of all ordinal numbers less than $\alpha$ is $\aleph_0$, and we may therefore write the elements of this set as the terms of an infinite sequence

$$\rho_0(\alpha), \rho_1(\alpha), \rho_2(\alpha), \cdots. \tag{2}$$

Similarly, we may rewrite the set of all terms in (1) which precede $s_\alpha$ as an infinite sequence

$$z_0(\alpha), z_1(\alpha), z_2(\alpha), \cdots, \tag{3}$$

where, for every $i=0, 1, 2, \cdots, z_i(\alpha)$ is an increasing infinite sequence of non-negative integers $(m_0^i(\alpha), m_1^i(\alpha), m_2^i(\alpha), \cdots)$. It is not difficult to show (cf. [**3**, Lemma I]; we shall not reproduce the proof here) that there exists a double infinite sequence of distinct non-negative integers $r_j^i(\alpha)$ $(i=0, 1, 2, \cdots; j=0, 1, 2, \cdots)$ such that, for every $i$, the terms of the infinite sequence $r_0^i(\alpha), r_1^i(\alpha), r_2^i(\alpha), \cdots$ all belong to the sequence $m_0^i(\alpha), m_1^i(\alpha), m_2^i(\alpha), \cdots$. Put

$$\xi_{r_j^i}^\alpha = \rho_j(\alpha) \text{ for every } i = 0, 1, 2, \cdots \text{ and every } j = 0, 1, 2, \cdots, \tag{4}$$

and $\xi_n^\alpha=0$ for every non-negative integer $n$ such that $n \neq r_j^i(\alpha)$ $(i=0, 1, 2, \cdots; j=0, 1, 2, \cdots)$.

Let $x$ be a given real number. Then there exists a unique ordinal number $\alpha$, with $\omega \leqq \alpha < \Omega$, such that $x=x_\alpha$. For every non-negative integer $k$, put

$$f_k(x) = t_k^{\xi_k^\alpha}. \tag{5}$$

The functions $f_k(x)$ $(k=0, 1, 2, \cdots)$ are thus defined for every real number $x$; we assert that they satisfy the conclusion of Theorem 1.

For let $N$ be a nondenumerable set of real numbers, and suppose that there are infinitely many functions of the sequence $f_0(x), f_1(x), f_2(x), \cdots$, each of which does not transform $N$ onto the set of all real numbers in such a manner that every real number is the image of nondenumerably many elements of $N$. Then there exists an increasing infinite sequence of non-negative integers $m_0, m_1, m_2, \cdots$ and an infinite sequence of real numbers $y_0, y_1, y_2, \cdots$ such that

(6) $y_k = f_{m_k}(x)$ for at most denumerably many $x \in N$ $(k=0, 1, 2, \cdots)$.

Now because of the correspondence mentioned at the beginning of the proof, there exists a $\mu < \Omega$ such that

$$y_k = t^{\mu}_{m_k} \qquad (k = 0, 1, 2, \cdots). \tag{7}$$

There is also a unique $\beta < \Omega$ for which $(m_0, m_1, m_2, \cdots) = s_\beta$.

Let $\alpha$ be an ordinal number such that

$$\max(\mu, \beta, \omega) < \alpha < \Omega. \tag{8}$$

According to the definition of (2), there exists a $j$ such that $\mu = \rho_j(\alpha)$. By the definition of (3), there exists an $i$ such that $s_\beta = z_i(\alpha)$. The sequence $r^i_0(\alpha), r^i_1(\alpha), r^i_2(\alpha), \cdots$, as specified above, has the property that, for some $k$, $r^i_j(\alpha) = m^i_k(\alpha) = m_k$. According to (4), therefore, $\mu = \xi^\alpha_{m_k}$. Hence, on account of (5) and (7), $f_{m_k}(x_\alpha) = y_k$. Since this holds for every $\alpha$ satisfying (8) ($k$ depending on $\alpha$), it follows, if we note (6), that $N$ is at most denumerable, contrary to our assumption. This completes the proof of Theorem 1. (The converse of Theorem 1 is also true, and is much easier to prove; see, e.g., [**5**, p. 14].)

We remark that an analogous argument can be employed to derive the following result, a weaker form of which is stated in [**1**, pp. 6–7]:

THEOREM 2. *Let $\alpha$ be an ordinal number, and $M$ be a set with $|M| = 2^{\aleph_\alpha}$. Then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ if, and only if, there exists a transfinite sequence of single-valued functions $f_0(x), f_1(x), \cdots, f_\tau(x), \cdots$ $(\tau < \omega_\alpha)$ defined for all $x \in M$, such that, for every subset $N$ of $M$ with $|N| > \aleph_\alpha$, every function of the sequence, except perhaps a number less than $\aleph_\alpha$ of them, transforms $N$ onto $M$ in such a manner that every element of $M$ is the image of more than $\aleph_\alpha$ elements of $N$.*

## REFERENCES

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